Lab Exercises for Lecture 1

Problem 1.1. Consider a well-founded strict ordering $\succ$ on atoms. Prove that the induced ordering on literals, as defined in the lecture, is also well-founded.

Problem 1.2. Consider an ordering $\succ$ on ground non-equality atoms that is total and well-founded. We denote the literal ordering induced by $\succ$ also by $\succ$. Let C and D be ground clauses without equality literals. Let A and B respectively denote the maximal atoms of C and D wrt $\succ$.
Assume that A and B are syntactically the same atoms. Assume also that A occurs negatively in C but only positively in D. Show that $C \succ_{bag} D$.

Problem 1.3. Consider strict partial orderings $\succ_i$ over M_i, for $i = 1, 2$. Assume that $\succ_1$ and $\succ_2$ are well-founded. We define the ordering $\succ^*$ over $M_1 \times M_2$ as:

$$(a_1, a_2) \succ^* (b_1, b_2) \Leftrightarrow \Big(a_1 \succ_1 b_1 \quad \text{or} \quad (a_1 = b_1 \quad \text{and} \quad a_2 \succ_2 b_2)\Big)$$

Show that $\succ^*$ is well-founded.

Problem 1.4. Let $\mathbb{I}$ be a sound inference system on clauses and let S_0 be a non-empty set of clauses. Consider a fair $\mathbb{I}$-inference process $S_0 \to S_1 \to S_2 \to \ldots$, without redundancy elimination. Let I_∞ denote the limit of this fair $\mathbb{I}$-inference process. Show that I_∞ is the $\mathbb{I}$-closure of S_0.

Note: You need to prove that I_∞ is the smallest $\mathbb{I}$-saturated set containing S_0. Recall and use the property from the lecture on $\mathbb{I}$-inference processes $S_0 \to S_1 \to S_2 \to \ldots$, in particular that every S_i is a subset of the $\mathbb{I}$-closure of S_0.

Problem 1.5. Let S be the following set of clauses:

$$\{\, \neg p \vee \neg q, \quad \neg p \vee q, \quad p \vee \neg q, \quad p \vee q \,\}$$

Consider the binary resolution inference system BR (without ordering and selection function). Show that there exists an infinite number of different BR derivations of the empty clause from the clauses of S.

Lab Exercises for Lecture 2

Problem 2.1. Let $\succ$ be a total well-founded ordering on the ground atoms $p_1, \ldots, p_6$ such that $p_6 \succ p_5 \succ p_4 \succ p_3 \succ p_2 \succ p_1$. Consider the bag extension of $\succ$; for simplicity, denote the bag extension of $\succ$ also by $\succ$.
Using $\succ$, compare and order the following three clauses:

$$p_6 \vee \neg p_6, \qquad \neg p_2 \vee p_4 \vee p_5, \qquad p_2 \vee p_3.$$

Problem 2.2. Let p, q be boolean atoms and let S be the following set of ground formulas:

$$\{\, \neg p \vee \neg q, \quad \neg p \vee q, \quad p \vee \neg q, \quad p \vee q \,\}$$

Take any ordering such that $p \succ q$ and any selection function σ over S such that

$$\{\, \neg p \vee \underline{\neg q}, \quad \underline{\neg p} \vee q, \quad p \vee \underline{\neg q}, \quad \underline{p} \vee q \,\}.$$

(a) Is σ a well-behaved selection function over S? Justify your answer!

(b) How many inferences of $\mathbb{BR}_\sigma$ are applicable to S? Justify your answer!

Problem 2.3. Give an example of a non-tautology ground clause with at least one selected literal so that this selection is not well-behaved for any ordering $\succ$. Justify your solution!

Problem 2.4. Let S be the set of clauses

$$\neg q \vee r, \quad \neg p \vee q, \quad \neg r \vee \neg q, \quad \neg q \vee \neg p, \quad \neg p \vee \neg r, \quad \neg r \vee p, \quad r \vee q \vee p$$

(a) Prove unsatisfiabiliy of S using BR.

(b) Formalize S in TPTP and prove its unsatisfiability using `Vampire`, by running `Vampire` with the additional option `-av off` .

Lab Exercises for Lecture 3

Problem 3.1. Consider a KBO ordering $\succ$ such that $inverse \gg times$ by precedence. Consider the literal:

$$inverse(times(x, y)) = times(inverse(y), inverse(x)).$$

Compare, w.r.t $\succ$, the left- and right-hand side terms of the equality when:

- $weight(inverse) = weigth(times) = 1$;
- $weight(inverse) = 0$ and $weight(times) = 1$.

Problem 3.2. Let Σ be a signature containing only function symbols such that Σ contains at least one constant. Let $\gg$ be a precedence relation on Σ and $w : \Sigma \to \mathbb{N}$ be a weight function compatible with $\gg$. Consider the (ground) Knuth-Bendix order $\succ$ induced by $\gg$ and w on the set of ground terms of Σ. Describe the set of ground terms that have the minimal weight wrt $\succ$.

Problem 3.3. Consider the set S of ground formulas:

$$\begin{aligned}\{\ & g(f(a)) = a \vee g(f(b)) = a,\\ & f(a) = a,\\ & f(b) \neq f(b) \vee f(b) = a,\\ & g(a) \neq a\ \}\end{aligned}$$

Show that S is unsatisfiable by applying saturation on S using an inference process based on the ground superposition calculus $\mathbb{S}\mathrm{up}_{\succ,\sigma}$ (including the inference rules of binary resolution $\mathbb{BR}_\sigma$), where σ is a well-behaved selection function wrt $\succ$ and:

(a) the ordering $\succ$ is the KBO ordering generated by the precedence $f \gg a \gg g \gg b$ and the weight function w with $w(f) = 0, w(b) = 1, w(a) = 2, w(g) = 3$;

(b) the ordering $\succ$ is the KBO ordering generated by the precedence $g \gg a \gg b \gg f$ and the weight function w with $w(g) = 0, w(b) = 1, w(f) = 1, w(a) = 3$.

Give details on what literals are selected and which terms are maximal.

Lab Exercises for Lecture 4

Problem 4.1. Apply the unification algorithm and show the most general unifier of the following atoms:

(a) $p(a, f(y), y)$ and $p(a, x, f(x))$;

(b) $p(f(x, a), f(f(b, a)))$ and $p(z, f(z))$;

(c) $p(f(x, y), f(y, z))$ and $p(z, f(w, f(y, w)))$.

Note: x, y, z, w denote variables, f is a function symbol, p is a predicate symbol and a, b are constants.

Problem 4.2. Consider the following set S of clauses:

$$\begin{array}{l} \neg p(z, a) \vee \neg p(z, x) \vee \neg p(x, z) \\ p(y, a) \vee p(y, f(y)) \\ p(w, a) \vee p(f(w), w) \end{array}$$

where p is a predicate symbol, f is a function symbol, x, y, z, w are variables and a is a constant.
Give a refutation proof of S by using the non-ground binary resolution inference system $\mathbb{BR}$. For each newly derived clause, label the clauses from which it was derived by which inference rule and indicate most general unifiers.

Problem 4.3. Let p denote a unary predicate symbol, f a unary function symbol, x, y variables and c a constant. Let C_1 be the clause $p(x) \vee p(y)$ and consider C_2 to be the clause $p(x)$. Further, let D denote the clause $p(f(c))$.

(a) Does C_1 subsume D?

(b) Does C_2 subsume D?

Justify your answers!

Problem 4.4. Let x denote a variable, a, b, c constants, and f a unary function symbol.
Give a superposition refutation of the following set of two clauses:

$$\{ \quad x = f(c), \quad a \neq b \quad \}$$

such that, in every inference, the premises and the conclusion of that inference do not use the symbols f, c together with the symbols a, b. That is, every inference has the following property: if the premise or the conclusion contain any of the symbols f, c, then the premise and the conclusion contain neither a nor b.
In your proof, use only the inference system of the superposition calculus $\mathbb{S}\text{up}$ (without ordering and selection function); that is, no inferences of binary resolution $\mathbb{BR}$ should be used. For each newly derived clause, clearly label the clauses from which it was derived and indicate most general unifiers.

Problem 4.5.
Let f be a unary and g be a binary function symbol. Further let a, b, c be constants, and x, y, z be variables. We define the weight function $w(s) = w(v) = 1$, for every symbol s and variable v, and let $g \gg f \gg c \gg b \gg a$. Answer the following questions using a KBO with the weight function w and the precedence relation $\gg$ to order terms, and its extension to compare literals and clauses.

(a) Do the clauses C_1 and C_2 make the clause C_3 redundant?

$$\begin{aligned} C_1 :&\ \ a \neq b \vee f(a) \neq a \\ C_2 :&\ \ f(f(x)) = a \\ C_3 :&\ \ f(f(b)) \neq b \vee f(a) \neq a \end{aligned}$$

(b) Does the clause C_4 make the clause C_5 redundant?

$$\begin{aligned} C_4 :&\ \ f(g(x, a)) \neq f(y) \\ C_5 :&\ \ f(g(x, z)) \neq f(g(y, b)) \vee f(g(x, b)) \neq f(g(a, b)) \end{aligned}$$

(c) Does the clause C_6 make the clause C_7 redundant?

$$\begin{aligned} C_6 :&\ \ g(x, y) \neq f(x) \\ C_7 :&\ \ g(f(x), f(z)) \neq f(f(x)) \vee g(a, b) \neq c \end{aligned}$$

Problem 4.6. Consider the following inference:

$$\frac{x = f(c) \vee p(x) \quad f(h(b)) = h(g(y, y)) \vee h(g(d, b)) \neq f(c)}{p(h(g(d, b))) \vee f(h(b)) = h(g(y, y))}$$

in the non-ground superposition inference system $\mathbb{S}\mathrm{up}$ (**without** the rules of the non-ground binary resolution inference system $\mathbb{BR}$), where p is a predicate symbol, f, g, h are function symbols, b, c, d are constants, and x, y are variables.

(a) Prove that the above inference is a sound inference of $\mathbb{S}\mathrm{up}$.

(b) Is the above inference a simplifying inference of $\mathbb{S}\mathrm{up}$? Justify your answer.

Problem 4.7. Recall that the inverse of the binary relation $r_1(x, y)$ is the binary relation $r_2(y, x)$ such that $r_1(x, y)$ if and only if $r_2(y, x)$.

Prove that the inverse of a dense order is also dense. For doing so, you are required to do the following steps:

- Formalize the problem in `TPTP` and prove it using `Vampire`.
- Explain the superposition reasoning part of the `Vampire` proof by detailing the superposition inferences, generated clauses and mgus in the poof. Use `Vampire` with the AVATAR option off, that is `-av off`.

Problem 4.8. Consider the group theory axiomatization used in the lecture. Prove that the group's left identity element e is also a right identity.

- Formalize the problem in `TPTP` and use it using `Vampire`, by running `Vampire` with the additional option `-av off` .
- Explain the superposition reasoning part of the `Vampire` proof by detailing the superposition inferences, generated clauses and mgus in the poof.