Satisfiability Modulo Theories

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Acknowledgments: Many thanks to Cesare Tinelli and Albert Oliveras for contributing some of the material used in these slides.

Disclamer: The literature on SMT and its applications is vast. The bibliographic references provided here are just a sample. Apologies to all authors whose work is not cited.

Introduction

A (Very) Brief History of Automated Reasoning

Philosophers have long dreamed of machines that can reason. The pursuit of this dream has occupied some of the best minds and led both to great acheivements and great disappointments.

~1700

Leibniz – mechanized human reasoning

1928

Hilbert

Entscheidungs-problem

1936

Church – lamda calculus

Turing – reduction halting problem

1954

Davis – decision procedure for Presburger arithmetic

Automated Reasoning: A Failure?

- At the turn of the century, automated reasoning was still considered by many to be impractical for most real-world applications
- Interesting problems appeared to be beyond the reach of automated methods because of decidability and complexity barriers
- The dream of *Hilbert*'s mechanized mathematics or *Leibniz*'s calculating machine was believed by many to be simply unattainable

The Satisfiability Revolution

Princeton, c. 2000

- *Chaff SAT solver*: orders of magnitude faster than previous SAT solvers
- *Important observation*: many real-world problems do not exhibit worst-case theoretical performance

Palo Alto, c. 2001

- Idea: combine fast new SAT solvers with decision procedures for decidable first-order theories
- *SVC*, *CVC* solvers (Stanford); *ICS*, *Yices* solvers (SRI)
- *Satisfiability Modulo Theories* (SMT) was born

SMT solvers

SMT solvers: *general-purpose* logic engines

- Given condition X, is it possible for Y to happen
- X and Y are expressed in a *rich logical language*
  - First-order logic
  - Domain-specific reasoning
    - arithmetic, arrays, bit-vectors, data types, etc.

SMT solvers are *changing the way people solve problems*

- Instead of building a *special-purpose* solver
- *Translate* into a logical formula and use an SMT solver
- Not only easier, *often better*

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A Security Example

Django

- Widely used open-source web development platform
- A security vulnerability in Django (CVE-2013-6044) was blamed on the following function[1]

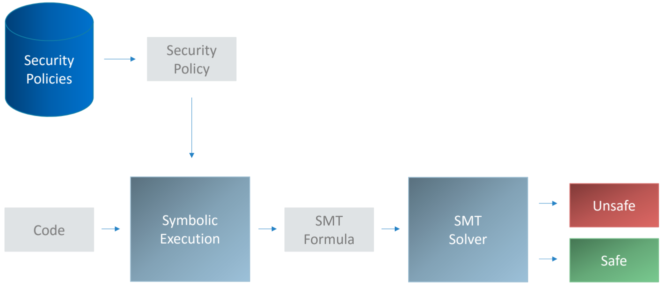
```
def is_safe_url(url, host=None):
    """
    Return ``True`` if the url is a safe redirection (i.e. it doesn't
    point to a different host).

    Always returns ``False`` on an empty url.
    """
    if not url:
        return False
    netloc = urllib_parse.urlparse(url)[1]
    return not netloc or netloc == host
```

[1] https://github.com/django/django/blob/09a5f5aabe27f63ec8d8982efa6cef9bf7b86022/django/utils/http.py#L252

Using SMT To Find the Vulnerability

An approach for finding security vulnerabilities

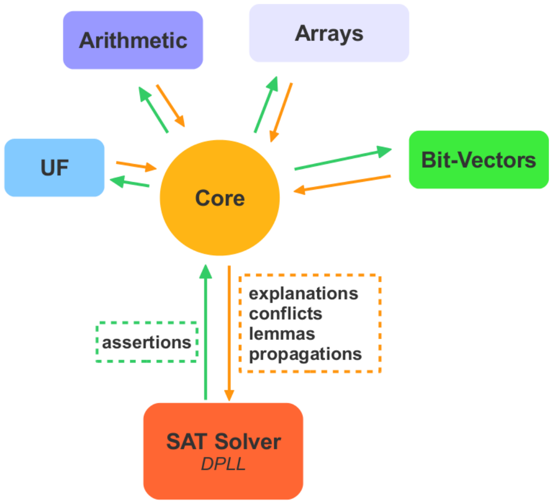
- *Symbolic execution*: generates a logical formula satisfiable iff code can violate security policy
- *SMT solver*: returns a solution or proves that none exists



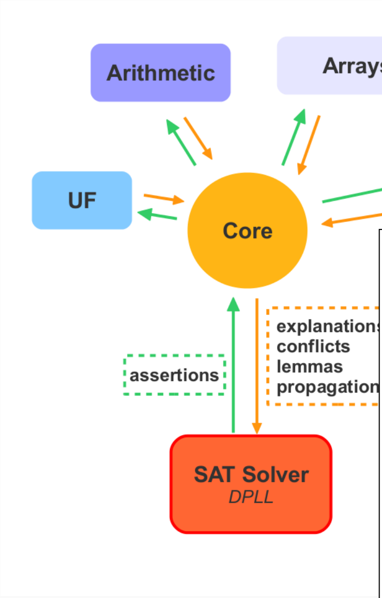
Using SMT To Find the Vulnerability

Demo: *Django XSS attack*

SMT Solvers


Arithmetic
Arrays
UF
Core
Bit-Vectors
assertions
explanations
conflicts
lemmas
propagations
SAT Solver
DPLL

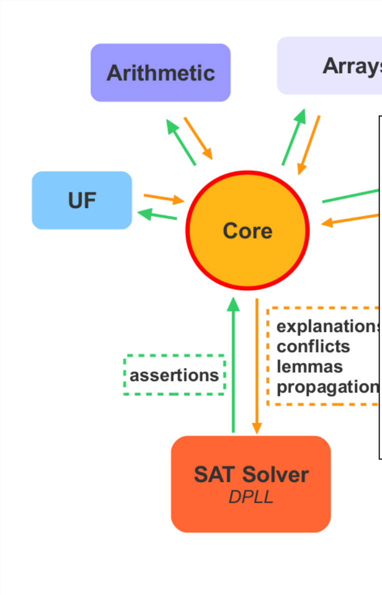
SMT Solvers



SAT Solver

- Only sees *Boolean skeleton* of problem
- Builds partial model by assigning truth values to literals
- Sends these literals to the core as *assertions*

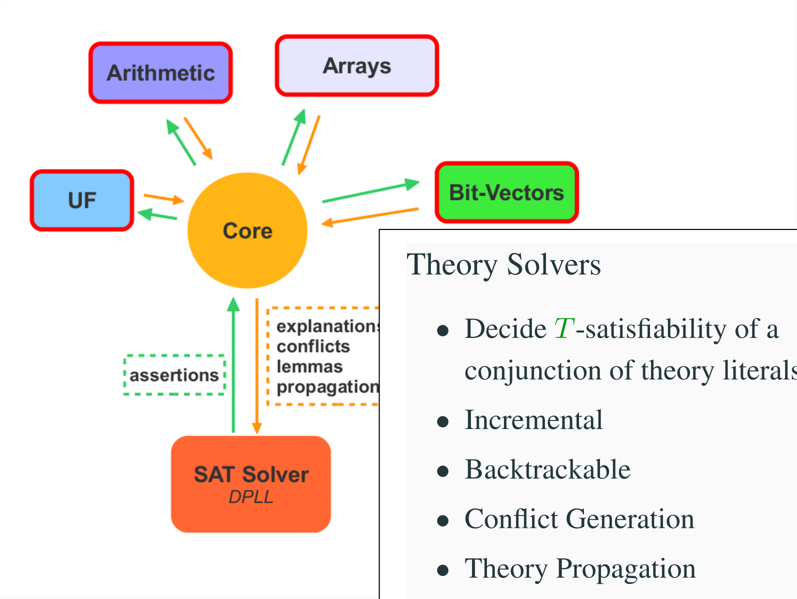
SMT Solvers



Core

- Sends each assertion to the appropriate theory
- Sends deduced literals to other theories/SAT solver
- Handles *theory combination*

SMT Solvers



Theory Solvers

- Decide T-satisfiability of a conjunction of theory literals
- Incremental
- Backtrackable
- Conflict Generation
- Theory Propagation

Theory Solvers

Theory Solvers

Given a theory T, a *Theory Solver* for T takes as input a set Φ of literals and determines whether Φ is T-satisfiable.

Φ is T-satisfiable iff there is some model M of T such that each formula in Φ holds in M.

Theories of Interest: UF

Equality ($=$) with Uninterpreted Functions [NO80, BD94, NO07]

Typically used to abstract unsupported constructs, e.g.:

- non-linear multiplication in arithmetic
- ALUs in circuits

Example: The formula

$$a * (|b| + c) = d \ \ \wedge \ \ b * (|a| + c) \neq d \ \ \wedge \ \ a = b$$

is unsatisfiable, but no arithmetic reasoning is needed

if we abstract it to

$$mul(a, add(abs(b), c)) = d \ \ \wedge \ \ mul(b, add(abs(a), c)) \neq d \ \ \wedge \ \ a = b$$

Theories of Interest: Arithmetic

Very useful, for obvious reasons

Restricted fragments (over the reals or the integers) support more efficient methods:

- Bounds: $x \bowtie k$ with $\bowtie \in \{<, >, \leq, \geq, =\}$ [BBC+05a]
- Difference logic: $x - y \bowtie k$, with $\bowtie \in \{<, >, \leq, \geq, =\}$ [NO05, WIGG05, CM06]
- UTVPI: $\pm x \pm y \bowtie k$, with $\bowtie \in \{<, >, \leq, \geq, =\}$ [LM05]
- Linear arithmetic, e.g: $2x - 3y + 4z \leq 5$ [DdM06]
- Non-linear arithmetic, e.g: $2xy + 4xz^2 - 5y \leq 10$ [BLNM+09, ZM10, JdM12]

Theories of Interest: Arrays

Used in software verification and hardware verification (for memories) [SBDL01, BNO$^+$08a, dMB09]

Two interpreted function symbols read and write

Axiomatized by:

- $\forall a\, \forall i\, \forall v\ \mathrm{read}(\mathrm{write}(a, i, v), i) = v$
- $\forall a\, \forall i\, \forall j\, \forall v\ i \neq j \rightarrow \mathrm{read}(\mathrm{write}(a, i, v), j) = \mathrm{read}(a, j)$

Sometimes also with *extensionality*:

- $\forall a\, \forall b\ (\forall i\, \mathrm{read}(a, i) = \mathrm{read}(b, i)\ \rightarrow a = b)$

Is the following set of literals satisfiable in this theory?

$$\mathrm{write}(a, i, x) \neq b,\ \mathrm{read}(b, i) = y,\ \mathrm{read}(\mathrm{write}(b, i, x), j) = y,\ a = b,\ i = j$$

Theories of Interest: Bit vectors

Useful both in hardware and software verification [BCF+07, BB09, HBJ+14]

Universe consists of (fixed-sized) vectors of bits

Different types of operations:

- *String-like*: concat, extract, …
- *Logical*: bit-wise not, or, and, …
- *Arithmetic*: add, subtract, multiply, …
- *Comparison*: $<, >, \ldots$

Is this formula satisfiable over bit vectors of size 3?

$$a[1:0] \neq b[1:0] \ \land \ (a \mid b) = c \ \land \ c[0] = 0 \ \land \ a[1] + b[1] = 0$$

Implementing a Theory Solver: Difference Logic

We consider a simple example: difference logic.

In *difference logic*, we are interested in the satisfiability of a conjunction of arithmetic atoms.

Each atom is of the form $x - y \bowtie c$, where x and y are variables, c is a numeric constant, and $\bowtie \in \{=, <, \leq, >, \geq\}$.

The variables can range over either the *integers* (QF_IDL) or the *reals* (QF_RDL).

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- $x - y = c \implies x - y \leq c \wedge x - y \geq c$
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- $x - y < c \implies x - y \leq c - 1$ (integers)
- $x - y < c \implies x - y \leq c - \delta$ (reals)

Difference Logic

Now we have a conjunction of literals, all of the form $x - y \leq c$.

From these literals, we form a weighted directed graph with a vertex for each variable.

For each literal $x - y \leq c$, there is an edge $x \xrightarrow{c} y$.

The set of literals is satisfiable iff there is no cycle for which the sum of the weights on the edges is negative.

There are a number of efficient algorithms for detecting negative cycles in graphs.

Difference Logic Example

$$x - y = 5 \ \wedge \ z - y \geq 2 \ \wedge \ z - x > 2 \ \wedge \ w - x = 2 \ \wedge \ z - w < 0$$

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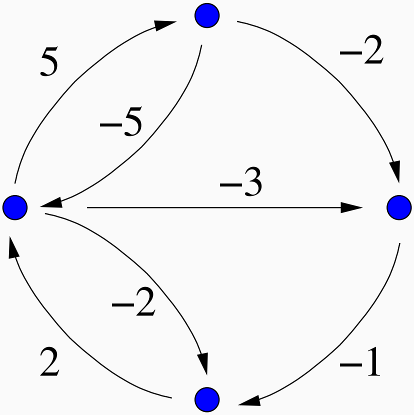
$$\begin{array}{lcl}
x - y = 5 & & x - y \leq 5 \wedge y - x \leq -5 \\
z - y \geq 2 & & y - z \leq -2 \\
z - x > 2 & \Rightarrow & x - z \leq -3 \\
w - x = 2 & & w - x \leq 2 \wedge x - w \leq -2 \\
z - w < 0 & & z - w \leq -1
\end{array}$$

Difference Logic Example



DPLL(T): Combining T-Solvers with SAT

Satisfiability Modulo a Theory T

Def. A formula is *(un)satisfiable in* a theory T, or T-*(un)satisfiable*, if there is a (no) model of T that satisfies it

Note: The T-satisfiability of quantifier-free formulas is decidable iff the T-satisfiability of conjunctions/sets of literals is decidable

(Convert the formula in DNF and check if any of its disjuncts is T-sat)

Problem: In practice, dealing with Boolean combinations of literals is as hard as in propositional logic

Solution: Exploit propositional satisfiability technology

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Lifting SAT Technology to SMT

Two main approaches:

1. "Eager" [PRSS99, SSB02, SLB03, BGV01, BV02]
   - translate into an equisatisfiable propositional formula
   - feed it to any SAT solver

Notable systems: *UCLID*

2. "Lazy" [ACG00, dMR02, BDS02, ABC+02]
   - abstract the input formula to a propositional one
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   - use a theory decision procedure to refine the formula and guide the SAT solver

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(Very) Lazy Approach for SMT – Example

$$g(a) = c \quad \land \quad f(g(a)) \neq f(c) \lor g(a) = d \quad \land \quad c \neq d$$

Theory T**:** Equality with Uninterpreted Functions

Simplest setting:

- Off-line SAT solver
- Non-incremental *theory solver* for conjunctions of equalities and disequalities
- Theory atoms (e.g., $g(a) = c$) abstracted to propositional atoms (e.g., 1)

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- SAT solver returns model $\{1,\ \overline{2},\ \overline{4}\}$.
  Theory solver finds (concretization of) $\{1,\ \overline{2},\ \overline{4}\}$ unsat.
- Send $\{1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4\}$ to SAT solver.
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- SAT solver finds $\{1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4\}$ unsat.
  Done: the original formula is unsatisfiable in UF.

Lazy Approach – Enhancements

Several enhancements are possible to increase efficiency:

- Check T-satisfiability only of full propositional model
- Check T-satisfiability of partial assignment M as it grows
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- If M is T-unsatisfiable, identify a T-unsatisfiable subset M_0 of M and add $\neg M_0$ as a clause
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Lazy Approach – Main Benefits

- Every tool does what it is good at:
  - SAT solver takes care of Boolean information
  - Theory solver takes care of theory information
- The theory solver works only with conjunctions of literals
- Modular approach:
  - SAT and theory solvers communicate via a simple API [GHN+04]
  - SMT for a new theory only requires new theory solver
  - An off-the-shelf SAT solver can be embedded in a lazy SMT system with few new lines of code (tens)

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An Abstract Framework for Lazy SMT

Several variants and enhancements of lazy SMT solvers exist

They can be modeled abstractly and declaratively as *transition systems*

A transition system is a binary relation over states, induced by a set of conditional transition rules

The framework can be first developed for SAT and then extended to lazy SMT [NOT06, KG07]

Advantages of Abstract Framework

An abstract framework helps one:

- skip over implementation details and unimportant control aspects
- reason formally about solvers for SAT and SMT
- model advanced features such as non-chronological bactracking, lemma learning, theory propagation, …
- describe different strategies and prove their correctness
- compare different systems at a higher level
- get new insights for further enhancements

The one described next is a re-elaboration of those in [NOT06, KG07]

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The Original DPLL Procedure

- Modern SAT solvers are based on the DPLL procedure [DP60, DLL62]
- DPLL tries to build incrementally a satisfying truth assignment M for a CNF formula F
- M is grown by
  - deducing the truth value of a literal from M and F, or
  - guessing a truth value
- If a wrong guess for a literal leads to an inconsistency, the procedure backtracks and tries the opposite value

An Abstract Framework for DPLL

States:

$$\mathsf{fail} \quad \text{or} \quad \langle M, F \rangle$$

where

- M is a sequence of literals and *decision points* $\bullet$ denoting a partial truth *assignment*
- F is a set of clauses denoting a CNF *formula*

Def. If $M = M_0 \bullet M_1 \bullet \cdots \bullet M_n$ where each M_i contains no decision points

- M_i is *decision level* i of M
- $M^{[i]} \stackrel{\text{def}}{=} M_0 \bullet \cdots \bullet M_i$

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Initial state:

- $\langle (), F_0 \rangle$, where F_0 is to be checked for satisfiability

Expected final states:

- fail if F_0 is unsatisfiable
- $\langle M, G \rangle$ otherwise, where
  - G is equivalent to F_0 and
  - M satisfies G

Transition Rules: Notation

States treated like records:

- M denotes the truth assignment component of current state
- F denotes the formula component of current state

Transition rules in *guarded assignment form* [KG07]

$$\frac{p_1 \quad \cdots \quad p_n}{[\mathsf{M} := e_1] \quad [\mathsf{F} := e_2]}$$

updating M, F or both when premises $p_1, \ldots, p_n$ all hold

Transition Rules for the Original DPLL

Extending the assignment

Propagate $$\frac{l_1 \vee \cdots \vee l_n \vee l \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\, l}$$

Note: When convenient, treat M as a set

Note: Clauses are treated modulo ACI of $\vee$

Decide $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M} \bullet l}$$

Note: $\mathrm{Lit}(F) \stackrel{\mathrm{def}}{=} \{l \mid l \text{ literal of } F\} \cup \{\bar{l} \mid l \text{ literal of } F\}$

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Transition Rules for the Original DPLL

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Backtrack

$$\frac{l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad \mathsf{M} = M \bullet l\, N \quad \bullet \notin N}{\mathsf{M} := M\, \bar{l}}$$

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From DPLL to CDCL Solvers (1)

To model conflict-driven backjumping and learning, add to states a third component C whose value is either no or a *conflict clause*

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From DPLL to CDCL Solvers (2)

Replace **Backtrack** with

Conflict $$\frac{\mathsf{C} = \mathsf{no} \quad l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M}}{\mathsf{C} := l_1 \vee \cdots \vee l_n}$$

Explain $$\frac{\mathsf{C} = l \vee D \quad l_1 \vee \cdots \vee l_n \vee \bar{l} \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Backjump $$\frac{\mathsf{C} = l_1 \vee \cdots \vee l_n \vee l \quad \mathsf{lev}\ \bar{l}_1, \ldots, \mathsf{lev}\ \bar{l}_n \leq i < \mathsf{lev}\ \bar{l}}{\mathsf{C} := \mathsf{no} \quad \mathsf{M} := \mathsf{M}^{[i]}\, l}$$

Maintain invariant: $\mathsf{F} \models_{\mathrm{p}} \mathsf{C}$ and $\mathsf{M} \models_{\mathrm{p}} \neg\mathsf{C}$ when $\mathsf{C} \neq \mathsf{no}$

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Note: $l \prec_{\mathsf{M}} l'$ if l occurs before l' in M
$\mathsf{lev}\ l = i$ iff l occurs in decision level i of M

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Execution Example

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $5 \vee 6$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\dots$			

Execution Example

$$F := \{1,\ \ \overline{1} \lor 2,\ \ \overline{3} \lor 4,\ \ \overline{5} \lor \overline{6},\ \ \overline{1} \lor \overline{5} \lor 7,\ \ \overline{2} \lor \overline{5} \lor 6 \lor \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \lor \overline{5} \lor 6 \lor \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \lor \overline{2} \lor \overline{5} \lor 6$	by **Explain** with $\overline{1} \lor \overline{5} \lor 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \lor \overline{2} \lor \overline{5}$	by **Explain** with $\overline{5} \lor \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\dots$			

Execution Example

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
...			

Execution Example

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

From DPLL to CDCL Solvers (4)

Also add

Learn $$\frac{\mathsf{F} \models_{\mathrm{p}} C \quad C \notin \mathsf{F}}{\mathsf{F} := \mathsf{F} \cup \{C\}}$$

Forget $$\frac{\mathsf{C} = \mathsf{no} \quad \mathsf{F} = G \cup \{C\} \quad G \models_{\mathrm{p}} C}{\mathsf{F} := G}$$

Restart $$\frac{}{\mathsf{M} := \mathsf{M}^{[0]} \quad \mathsf{C} := \mathsf{no}}$$

Note: Learn can be applied to any clause stored in C when $\mathsf{C} \neq \mathsf{no}$

Modeling Modern SAT Solvers

At the core, current CDCL SAT solvers are implementations of the transition system with rules

Propagate, **Decide**,

Conflict, **Explain**, **Backjump**,

Learn, **Forget**, **Restart**

Basic DPLL $\stackrel{\text{def}}{=}$

{ **Propagate**, **Decide**, **Conflict**, **Explain**, **Backjump** }

DPLL $\stackrel{\text{def}}{=}$ Basic DPLL + { **Learn**, **Forget**, **Restart** }

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The Basic DPLL System – Correctness

Some terminology:

Irreducible state: state for which no Basic DPLL rules apply

Execution: sequence of transitions allowed by the rules and starting with $\mathsf{M} = \emptyset$ and $\mathsf{C} = \mathsf{no}$

Exhausted execution: execution ending in an irreducible state

Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is unsatisfiable.

Proposition (Completeness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with $\mathsf{C} = \mathsf{no}$, the clause set F_0 is satisfied by M.

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Proposition (Strong Termination) Every execution in Basic DPLL is finite.

Note: This is not so immediate, because of **Backjump**.

Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is unsatisfiable.

Proposition (Completeness) For every exhausted execution starting

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Exhausted execution: execution ending in an irreducible state

Proposition (Strong Termination) Every execution in Basic DPLL is finite.

Lemma Every exhausted execution ends with either $\mathsf{C} = \mathsf{no}$ or fail.

Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is unsatisfiable.

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The DPLL System – Strategies

- Applying
  - one Basic DPLL rule between each two **Learn** applications and
  - **Restart** less and less often

  ensures termination

- A common basic strategy applies the rules with the following priorities:
  1. If $n > 0$ conflicts have been found so far, increase n and apply **Restart**
  2. If a clause is falsified by M, apply **Conflict**
  3. Keep applying **Explain** until **Backjump** is applicable
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  5. Apply **Backjump**
  6. Apply **Propagate** to completion
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     increase n and apply **Restart**
  2. If a clause is falsified by M, apply **Conflict**
  3. Keep applying **Explain** until **Backjump** is applicable
  4. Apply **Learn**
  5. Apply **Backjump**
  6. Apply **Propagate** to completion
  7. Apply **Decide**

From SAT to SMT

Same states and transitions but

- F contains quantifier-free clauses in some theory T
- M is a sequence of theory literals and decision points
- the DPLL system is augmented with rules

  T**-Conflict**, T**-Propagate**, T**-Explain**

- maintains invariant: $\mathsf{F} \models_T \mathsf{C}$ and $\mathsf{M} \models_{\mathrm{p}} \neg\mathsf{C}$ when $\mathsf{C} \neq \mathsf{no}$

Def. $F \models_T G$ iff every model of T that satisfies F satisfies G as well

SMT-level Rules

Fix a theory T

T-Conflict $$\frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_T \bot}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

T-Propagate $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad \mathsf{M} \models_T l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

T-Explain $$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_T \bar{l} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Note: $\bot$ = empty clause

Note: $\models_T$ decided by theory solver

SMT-level Rules

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Note: $\bot$ = empty clause

Note: $\models_T$ decided by theory solver

Modeling the Very Lazy Theory Approach

***T*-Conflict** is enough to model the naive integration of SAT solvers and theory solvers seen in the earlier UF example

$$\underbrace{g(a)=c}_{1} \quad \wedge \quad \underbrace{f(g(a))\neq f(c)}_{\overline{2}} \vee \underbrace{g(a)=d}_{3} \quad \wedge \quad \underbrace{c\neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	by **Propagate**$^{+}$
$1\ \overline{4}\bullet\overline{2}$	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4}\bullet\overline{2}$	$1,\ \overline{2}\vee 3,\ \overline{4}$	$\overline{1}\vee 2\vee 4$	by ***T*-Conflict**
$1\ \overline{4}\bullet\overline{2}$	$1,\ \overline{2}\vee 3,\ \overline{4},\ \overline{1}\vee 2\vee 4$	$\overline{1}\vee 2\vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2}\vee 3,\ \overline{4},\ \overline{1}\vee 2\vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2}\vee 3,\ \overline{4},\ \overline{1}\vee 2\vee 4$	no	by **Propagate**$^{+}$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2}\vee 3,\ \overline{4},\ \overline{1}\vee 2\vee 4,\ \overline{1}\vee\overline{3}\vee 4$	$\overline{1}\vee\overline{3}\vee 4$	by ***T*-Conflict**, **Learn**
fail			by **Fail**

Modeling the Very Lazy Theory Approach

$$\underbrace{g(a) = c}_{1} \quad \land \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \lor \underbrace{g(a) = d}_{3} \quad \land \quad \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	by **Propagate**$^{+}$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	$\overline{1} \lor 2 \lor 4$	by **T-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4},\ \overline{1} \lor 2 \lor 4$	$\overline{1} \lor 2 \lor 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \lor 3,\ \overline{4},\ \overline{1} \lor 2 \lor 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \lor 3,\ \overline{4},\ \overline{1} \lor 2 \lor 4$	no	by **Propagate**$^{+}$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \lor 3,\ \overline{4},\ \overline{1} \lor 2 \lor 4,\ \overline{1} \lor \overline{3} \lor 4$	$\overline{1} \lor \overline{3} \lor 4$	by **T-Conflict**, **Learn**
fail			by **Fail**

Modeling the Very Lazy Theory Approach

$$\underbrace{g(a) = c}_{1} \quad \land \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \lor \underbrace{g(a) = d}_{3} \quad \land \quad \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	$\overline{1} \lor 2 \lor 4$	by T**-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4},\ \overline{1} \lor 2 \lor 4$	$\overline{1} \lor 2 \lor 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \lor 3,\ \overline{4},\ \overline{1} \lor 2 \lor 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \lor 3,\ \overline{4},\ \overline{1} \lor 2 \lor 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \lor 3,\ \overline{4},\ \overline{1} \lor 2 \lor 4,\ \overline{1} \lor \overline{3} \lor 4$	$\overline{1} \lor \overline{3} \lor 4$	by T**-Conflict**, **Learn**
fail			by **Fail**

Modeling the Very Lazy Theory Approach

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^{+}$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by T**-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^{+}$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by T**-Conflict**, **Learn**
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$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by **T-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
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$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict**, **Learn**
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$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
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$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by ***T*-Conflict**, **Learn**
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A Better Lazy Approach

The very lazy approach can be improved considerably with

- An *on-line* SAT engine,
  which can accept new input clauses on the fly
- an *incremental and explicating* T-solver,
  which can
  1. check the T-satisfiability of M as it is extended and
  2. identify a small T-unsatisfiable subset of M once M becomes T-unsatisfiable

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Lazy Approach – Strategies

Ignoring **Restart** (for simplicity), a common strategy is to apply the rules using the following priorities:

1. If a clause is falsified by the current assignment M, apply **Conflict**
2. If M is T-unsatisfiable, apply T**-Conflict**
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Lazy Approach – Strategies

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Theory Propagation

With T**-Conflict** as the only theory rule, the theory solver is used just to validate the choices of the SAT engine

With T**-Propagate** and T**-Explain**, it can also be used to guide the engine's search [Tin02]

T**-Propagate** $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad \mathsf{M} \models_T l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

T**-Explain** $$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_T \bar{l} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

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$$\underbrace{g(a)=c}_{1} \quad \wedge \quad \underbrace{f(g(a))\neq f(c)}_{\overline{2}} \vee \underbrace{g(a)=d}_{3} \quad \wedge \quad \underbrace{c\neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2$	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	by T-**Propagate** $(1 \models_T 2)$
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Modeling Modern Lazy SMT Solvers

At the core, current lazy SMT solvers are implementations of the transition system with rules

(1) **Propagate**, **Decide**, **Conflict**, **Explain**, **Backjump**, **Fail**

(2) T**-Conflict**, T**-Propagate**, T**-Explain**

(3) **Learn**, **Forget**, **Restart**

Basic DPLL Modulo Theories $\stackrel{\text{def}}{=}$ (1) + (2)

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Correctness

Updated terminology:

Irreducible state: state to which no Basic DPLL MT rules apply

Execution: sequence of transitions allowed by the rules and starting with $\mathsf{M} = \emptyset$ and $\mathsf{C} = \mathsf{no}$

Exhausted execution: execution ending in an irreducible state

Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is T-unsatisfiable.

Proposition (Completeness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with $\mathsf{C} = \mathsf{no}$, F_0 is T-satisfiable; specifically, M is T-satisfiable and $\mathsf{M} \models_{\mathrm{p}} F_0$.

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Proposition (Termination) Every execution in which

(a) **Learn/Forget** are applied only finitely many times and

(b) **Restart** is applied with increased periodicity

is finite.

Lemma Every exhausted execution ends with either $\mathsf{C} = \mathsf{no}$ or fail.

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DPLL(T) Architecture

The approach formalized so far can be implemented with a simple architecture named DPLL(T) [GHN+04, NOT06]

$$\text{DPLL}(T) = \text{DPLL}(X) \text{ engine} + T\text{-solver}$$

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DPLL(T) = DPLL(X) engine + T-solver

DPLL(X):

- Very similar to a SAT solver, enumerates Boolean models
- Not allowed: pure literal, blocked literal detection, ...
- Required: incremental addition of clauses
- Desirable: partial model detection

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$$\text{DPLL}(T) = \text{DPLL}(X) \text{ engine} + T\text{-solver}$$

T-solver:

- Checks the T-satisfiability of conjunctions of literals
- Computes theory propagations
- Produces explanations of T-unsatisfiability/propagation
- Must be incremental and backtrackable

Reasoning by Cases in Theory Solvers

For certain theories, determining that a set M is T-unsatisfiable requires reasoning by cases.

Example: T = the theory of arrays.

$$M = \{\underbrace{r(w(a,i,x),j) \neq x}_{1},\ \underbrace{r(w(a,i,x),j) \neq r(a,j)}_{2}\}$$

$i = j$) Then, $r(w(a,i,x),j) = x$. Contradiction with 1.

$i \neq j$) Then, $r(w(a,i,x),j) = r(a,j)$. Contradiction with 2.

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Case Splitting

A *complete* T-solver reasons by cases via (internal) case splitting and backtracking mechanisms

An alternative is to lift case splitting and backtracking from the T-solver to the SAT engine

Basic idea: encode case splits as sets of clauses and send them as needed to the SAT engine for it to split on them [BNOT06]

Possible benefits:

- All case-splitting is coordinated by the SAT engine
- Only have to implement case-splitting infrastructure in one place
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Basic Scenario:

$$\mathsf{M} = \{\ldots,\ s = \underbrace{r(w(a,i,t),j)}_{s'},\ \ldots\}$$

- Main SMT module: "Is M $\mathcal{T}$-unsatisfiable?"
- $\mathcal{T}$-solver: "I do not know yet, but it will help me if you consider these *theory lemmas*:

$$s = s' \wedge i = j \rightarrow s = t, \quad s = s' \wedge i \neq j \rightarrow s = r(a,j)\text{"}$$

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To model the generation of theory lemmas for case splits, add the rule

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$$\frac{\models_T \exists \mathbf{v}(l_1 \lor \cdots \lor l_n) \quad l_1, \ldots, l_n \in L_\mathrm{S} \quad \mathbf{v} \text{ vars not in } \mathsf{F}}{\mathsf{F} := \mathsf{F} \cup \{l_1 \lor \cdots \lor l_n\}}$$

where L_S is a finite set of literals dependent on the initial set of clauses (see [BNOT06] for a formal definition of L_S)

Note: For many theories with a theory solver, there exists an appropriate finite L_S for every input F

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Now we can relax the requirement on the theory solver:

When $\mathsf{M} \models_{\mathrm{p}} \mathsf{F}$*, it must either*

- *determine whether* $\mathsf{M} \models_T \bot$ *or*
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The T-solver is required to determine whether $\mathsf{M} \models_T \bot$ only if all literals in L_{S} are defined in M

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Example — Theory of Finite Sets

$$F: \quad x = y \cup z \quad \wedge \quad y \neq \emptyset \vee x \neq z$$

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$x = y \cup z$	F	by **Propagate**$^+$
$x = y \cup z \;\bullet\; y = \emptyset$	F	by **Decide**
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$\mathcal{T}$-solver can make the following deductions at this point:

$$e \in x \;\; \cdots \;\; \Rightarrow \;\; e \in y \cup z \;\; \cdots \;\; \Rightarrow \;\; e \in y \;\; \cdots \;\; \Rightarrow \;\; e \in \emptyset \;\; \Rightarrow \;\; \bot$$

This enables an application of $\mathcal{T}$**-Conflict** with clause

$$x \neq y \cup z \;\vee\; y \neq \emptyset \;\vee\; x = z \;\vee\; e \notin x \;\vee\; e \in z$$

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$x = y \cup z$	F	by **Propagate**$^+$
$x = y \cup z \;\bullet\; y = \emptyset$	F	by **Decide**
$x = y \cup z \;\bullet\; y = \emptyset \; x \neq z$	F	by **Propagate**
$x = y \cup z \;\bullet\; y = \emptyset \; x \neq z$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by $\mathcal{T}$**-Learn**
$x = y \cup z \;\bullet\; y = \emptyset \; x \neq z \;\bullet\; e \in x$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by **Decide**
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$\mathcal{T}$-solver can make the following deductions at this point:

$$e \in x \quad \cdots \quad \Rightarrow \quad e \in y \cup z \quad \cdots \quad \Rightarrow \quad e \in y \quad \cdots \quad \Rightarrow \quad e \in \emptyset \quad \Rightarrow \quad \bot$$

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Correctness Results

Correctness results can be extended to the new rule.

Soundness: The new T**-Learn** rule maintains satisfiability of the clause set.

Completeness: As long as the theory solver can decide $\mathsf{M} \models_T \bot$ when all literals in L_S are determined, the system is still complete.

Termination: The system terminates under the same conditions as before. Roughly:

- Any lemma is (re)learned only finitely many times
- **Restart** is applied with increased periodicity

Combining Theories

Need for Combining Theories and Solvers

Recall: Many applications give rise to formulas like:

$$a \approx b + 2 \wedge A \approx \text{write}(B, a+1, 4) \wedge$$
$$(\text{read}(A, b+3) \approx 2 \vee f(a-1) \not\approx f(b+1))$$

Solving that formula requires reasoning over

- the theory of linear arithmetic (T_{LA})
- the theory of arrays (T_{A})
- the theory of uninterpreted functions (T_{UF})

Question: Given solvers for each theory, can we combine them modularly into one for $T_{\text{LA}} \cup T_{\text{A}} \cup T_{\text{UF}}$?

Under certain conditions, we can do it with the Nelson-Oppen combination method [NO79, Opp80]

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Motivating Example (Convex Case)

Consider the following set of literals over $T_{\mathrm{LRA}} \cup T_{\mathrm{UF}}$ (T_{LRA}, linear real arithmetic):

$$\begin{aligned} f(f(x) - f(y)) &= a \\ f(0) &> a + 2 \\ x &= y \end{aligned}$$

First step: *purify* literals so that each belongs to a single theory

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$$f(0) > a + 2 \quad \Longrightarrow \quad \begin{array}{c} f(e_4) > a + 2 \\ e_4 = 0 \end{array} \quad \Longrightarrow \quad \begin{array}{c} f(e_4) = e_5 \\ e_4 = 0 \\ e_5 > a + 2 \end{array}$$

Motivating Example (Convex Case)

Second step: exchange entailed *interface equalities*, equalities over shared constants $e_1, e_2, e_3, e_4, e_5, a$

L_1	L_2
$f(e_1) = a$	$e_2 - e_3 = e_1$
$f(x) = e_2$	$e_4 = 0$
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Third step: check for satisfiability locally

Report unsatisfiable

$L_2 \models_{\mathrm{LRA}} \bot$

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$L_1 \models_{\text{UF}} a = e_5$

Third step: check for satisfiability locally

$L_1 \not\models_{\text{UF}} \bot$

$L_2 \models_{\text{LRA}} \bot$

Report unsatisfiable

Motivating Example (Convex Case)

Second step: exchange entailed *interface equalities*, equalities over shared constants $e_1, e_2, e_3, e_4, e_5, a$

L_1	L_2
$f(e_1) = a$	$e_2 - e_3 = e_1$
$f(x) = e_2$	$e_4 = 0$
$f(y) = e_3$	$e_5 > a + 2$
$f(e_4) = e_5$	$e_2 = e_3$
$x = y$	$a = e_5$
$e_1 = e_4$	

$L_1 \models_{\mathrm{UF}} e_2 = e_3$ $\quad$ $L_2 \models_{\mathrm{LRA}} e_1 = e_4$

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Consider the following unsatisfiable set of literals over $T_{\text{LIA}} \cup T_{\text{UF}}$ (T_{LIA}, linear integer arithmetic):

$$\begin{aligned} 1 \leq \quad & x \quad \leq 2 \\ f(1) \quad &= \quad a \\ f(2) \quad &= \quad f(1) + 3 \\ a \quad &= \quad b + 2 \end{aligned}$$

First step: *purify* literals so that each belongs to a single theory

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$$f(1) = a \implies \begin{aligned} f(e_1) &= a \\ e_1 &= 1 \end{aligned}$$

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$$f(2) = f(1) + 3 \implies \begin{aligned} e_2 &= 2 \\ f(e_2) &= e_3 \\ f(e_1) &= e_4 \\ e_3 &= e_4 + 3 \end{aligned}$$

Motivating Example (Non-convex Case)

Second step: exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

L_1	L_2
$1 \leq x$	$f(e_1) = a$
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$e_1 = 1$	$f(e_2) = e_3$
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$e_2 = 2$	$x = e_1$
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No more entailed equalities, but $L_1 \models_{\text{LIA}} x = e_1 \vee x = e_2$

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$e_2 = 2$	$x = e_1$
$e_3 = e_4 + 3$	
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Consider each case of $x = e_1 \vee x = e_2$ separately

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Case 1) $x = e_1$

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$L_2 \models_{\text{UF}} a = b$, which entails $\perp$ when sent to L_1

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$e_3 = e_4 + 3$	
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$e_1 = 1$	$f(e_2) = e_3$
$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	$x = e_2$
$e_3 = e_4 + 3$	
$a = e_4$	
$x = e_2$	

Case 2) $x = e_2$

Motivating Example (Non-convex Case)

Second step: exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

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$x \leq 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
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$L_2 \models_{\text{UF}} e_3 = b$, which entails $\perp$ when sent to L_1

The Nelson-Oppen Method

- For $i = 1, 2$, let T_i be a first-order theory of *signature* Σ_i (set of function and predicate symbols in T_i other than $=$)
- Let $T = T_1 \cup T_2$
- Let $\mathcal{C}$ be a finite set of *free* constants (i.e., not in $\Sigma_1 \cup \Sigma_2$)

We consider only input problems of the form

$$L_1 \cup L_2$$

where each L_i is a finite set of *ground* (i.e., variable-free) $(\Sigma_i \cup \mathcal{C})$-literals

Note: Because of purification, there is no loss of generality in considering only ground $(\Sigma_i \cup \mathcal{C})$-literals

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The Nelson-Oppen Method

Bare-bones, non-deterministic, non-incremental version

[Opp80, Rin96, TH96]:

Input: $L_1 \cup L_2$ with L_i finite set of ground $(\Sigma_i \cup \mathcal{C})$-literals
Output: **sat** or **unsat**

1. Guess an *arrangement* A, i.e., a set of equalities and disequalities over $\mathcal{C}$ such that

$$c = d \in A \text{ or } c \neq d \in A \text{ for all } c, d \in \mathcal{C}$$

2. If $L_i \cup A$ is T_i-unsatisfiable for $i = 1$ or $i = 2$, return unsat
3. Otherwise, return **sat**

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Correctness of the NO Method

Proposition (Termination) The method is terminating.

(Trivially, because there is only a finite number of arrangements to guess)

Proposition (Soundness) If the method returns **unsat** for every arrangement, the input is $(T_1 \cup T_2)$-unsatisfiable.

(Because satisfiability in $(T_1 \cup T_2)$ is always preserved)

Proposition (Completeness) If $\Sigma_1 \cap \Sigma_2 = \emptyset$ and T_1 and T_2 are stably infinite, when the method returns **sat** for some arrangement, the input is $(T_1 \cup T_2)$-is satisfiable.

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Stably Infinite Theories

Def. A theory T is *stably infinite* iff every quantifier-free T-satisfiable formula is satisfiable in an infinite model of T

Many interesting theories are stably infinite:

- Theories of an infinite structure (e.g., integer arithmetic)
- Complete theories with an infinite model (e.g., theory of dense linear orders, theory of lists)
- Convex theories (e.g., EUF, linear real arithmetic)

Def. A theory T is *convex* iff, for any set L of literals
$L \models_T s_1 = t_1 \vee \cdots \vee s_n = t_n \implies L \models_T s_i = t_i$ for some i

Note: With convex theories, arrangements do not need to be guessed—they can be computed by (theory) propagation

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The Nelson-Oppen method has been extended to some classes of non-stably infinite theories [TZ05, RRZ05, JB10]

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SMT Solving with *Multiple* Theories

Let $T_1, \ldots, T_n$ be theories with respective solvers $S_1, \ldots, S_n$

How can we integrate all of them cooperatively into a single SMT solver for $T = T_1 \cup \cdots \cup T_n$?

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Quick Solution:

1. Combine $S_1, \ldots, S_n$ with Nelson-Oppen into a theory solver for T
2. Build a DPLL(T) solver as usual

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Better Solution [Bar02, BBC$^+$05b, BNOT06]:

1. Extend DPLL(T) to DPLL($T_1, \ldots, T_n$)
2. Lift Nelson-Oppen to the DPLL($X_1, \ldots, X_n$) level
3. Build a DPLL($T_1, \ldots, T_n$) solver

Modeling DPLL($T_1, \ldots, T_n$) Abstractly

- Let $n = 2$, for simplicity
- Let T_i be of signature Σ_i for $i = 1, 2$, with $\Sigma_1 \cap \Sigma_2 = \emptyset$
- Let $\mathcal{C}$ be a set of free constants
- Assume wlog that each input literal has signature $(\Sigma_1 \cup \mathcal{C})$ or $(\Sigma_2 \cup \mathcal{C})$ (no *mixed* literals)
- Let $\mathsf{M}|_i \stackrel{\text{def}}{=} \{(\Sigma_i \cup \mathcal{C})\text{-literals of } \mathsf{M} \text{ and their complement}\}$
- Let $\mathrm{I}(\mathsf{M}) \stackrel{\text{def}}{=} \{c = d \mid c, d \text{ occur in } \mathcal{C}, \mathsf{M}|_1 \text{ and } \mathsf{M}|_2\} \cup$
  $\{c \neq d \mid c, d \text{ occur in } \mathcal{C}, \mathsf{M}|_1 \text{ and } \mathsf{M}|_2\}$

  (*interface literals*)

Abstract DPLL Modulo Multiple Theories

Propagate, **Conflict**, **Explain**, **Backjump**, **Fail** (unchanged)

Decide $\dfrac{l \in \mathrm{Lit}(\mathsf{F}) \cup \mathrm{I}(\mathsf{M}) \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M} \bullet l}$

Only change: decide on interface equalities as well

T-Propagate $\dfrac{l \in \mathrm{Lit}(\mathsf{F}) \cup \mathrm{I}(\mathsf{M}) \quad i \in \{1, 2\} \quad \mathsf{M} \models_{T_i} l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$

Only change: propagate interface equalities as well, but reason locally in each T_i

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Abstract DPLL Modulo Multiple Theories

T-Conflict

$$\frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_{T_i} \bot \quad i \in \{1, 2\}}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

T-Explain

$$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_{T_i} \bar{l} \quad i \in \{1, 2\} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Only change: reason locally in each T_i

I-Learn

$$\frac{\models_{T_i} l_1 \vee \cdots \vee l_n \quad l_1, \ldots, l_n \in \mathsf{M}|_i \cup \mathrm{I}(\mathsf{M}) \quad i \in \{1, 2\}}{\mathsf{F} := \mathsf{F} \cup \{l_1 \vee \cdots \vee l_n\}}$$

New rule: for entailed disjunctions of interface literals

Abstract DPLL Modulo Multiple Theories

T-Conflict

$$\frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_{T_i} \bot \quad i \in \{1, 2\}}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

T-Explain

$$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_{T_i} \bar{l} \quad i \in \{1, 2\} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

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I-**Learn**

$$\frac{\models_{T_i} l_1 \vee \cdots \vee l_n \quad l_1, \ldots, l_n \in \mathsf{M}|_i \cup \mathrm{I}(\mathsf{M}) \quad i \in \{1, 2\}}{\mathsf{F} := \mathsf{F} \cup \{l_1 \vee \cdots \vee l_n\}}$$

New rule: for entailed disjunctions of interface literals

Example — Convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = e_2}^{1} \wedge \overbrace{f(y) = e_3}^{2} \wedge \overbrace{f(e_4) = e_5}^{3} \wedge \overbrace{x = y}^{4} \wedge$$
$$\underbrace{e_2 - e_3 = e_1}_{5} \wedge \underbrace{e_4 = 0}_{6} \wedge \underbrace{e_5 > a + 2}_{7}$$

$$\underbrace{e_2 = e_3}_{8} \quad \underbrace{e_1 = e_4}_{9} \quad \underbrace{a = e_5}_{10}$$

M	F	C	rule
	F	no	
0 1 2 3 4 5 6 7	F	no	by **Propagate**$^+$
0 1 2 3 4 5 6 7 8	F	no	by T-**Propagate** ($1, 2, 4 \models_{\text{UF}} 8$)
0 1 2 3 4 5 6 7 8 9	F	no	by T-**Propagate** ($5, 6, 8 \models_{\text{LRA}} 9$)
0 1 2 3 4 5 6 7 8 9 10	F	no	by T-**Propagate** ($0, 3, 9 \models_{\text{UF}} 10$)
0 1 2 3 4 5 6 7 8 9 10	F	$\overline{7} \vee \overline{10}$	by T-**Conflict** ($7, 10 \models_{\text{LRA}} \bot$)
fail			by **Fail**

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0 1 2 3 4 5 6 7 8 9 10	F	$\overline{7} \vee \overline{10}$	by T-**Conflict** $(7, 10 \models_{\text{LRA}} \bot)$
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Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
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$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T-**Propagate** $(0,\ \underline{3} \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T-**Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ \underline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T-**Conflict** $(7,\ 13 \models_{\mathrm{UF}} \perp)$
$0 \cdots 9\ \underline{10}\ \underline{\overline{13}}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ \underline{10}\ \underline{\overline{13}}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T-**Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

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$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
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$0 \cdots 9\ \underline{10}\ \underline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ \underline{10}\ \underline{13}\ \underline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{UF} \overline{11})$
$0 \cdots 9\ 10\ 13\ 11\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\ldots$			(exercise)
fail	$\ldots$	$\ldots$	by **Fail**

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$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ \overline{10}\ \overline{13}\ 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{UF} \overline{11})$
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$0 \cdots 9\ 10$	F	no	by ***T*-Propagate** $(0,\ 3 \models_{UF} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{LIA} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by ***T*-Propagate** $(0,\ 1,\ 11 \models_{UF} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by ***T*-Conflict** $(7,\ 13 \models_{UF} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by ***T*-Propagate** $(0,\ 1,\ \overline{13} \models_{UF} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$F := \underbrace{\overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge}_{} \; \underbrace{1 \le x}_{4} \wedge \underbrace{x \le 2}_{5} \wedge \underbrace{e_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{e_2 = 2}_{8} \wedge \underbrace{e_3 = e_4 + 3}_{9}$$

$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by ***T*-Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by ***T*-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by ***T*-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by ***T*-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
…			(exercise)
fail	…	…	by **Fail**

Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
$$\underbrace{1 \leq x}_{4} \wedge \underbrace{x \leq 2}_{5} \wedge \underbrace{e_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{e_2 = 2}_{8} \wedge \underbrace{e_3 = e_4 + 3}_{9}$$

$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by **T-Propagate** $(0,\ 3 \models_{\text{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\text{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ 11 \models_{\text{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by **T-Conflict** $(7,\ 13 \models_{\text{UF}} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ \overline{13} \models_{\text{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by **T-Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by **T-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by **T-Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by **T-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
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Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
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$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{LIA} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ 11 \models_{UF} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by **T-Conflict** $(7,\ 13 \models_{UF} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ \overline{13} \models_{UF} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$F := \underbrace{\overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge}{} $$
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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T-**Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by I-**Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T-**Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T-**Conflict** $(7,\ 13 \models_{\mathrm{UF}} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T-**Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Suggested Readings

1. R. Nieuwenhuis, A. Oliveras, and C. Tinelli. **Solving SAT and SAT Modulo Theories: From an abstract Davis-Putnam-Logemann- Loveland procedure to DPLL(T)**. Journal of the ACM, 53(6):937-977, 2006.

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