Preprocessing and Inprocessing

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The Satisfiability (SAT) problem

$$(x_5 \vee x_8 \vee \bar{x}_2) \wedge (x_2 \vee \bar{x}_1 \vee \bar{x}_3) \wedge (\bar{x}_8 \vee \bar{x}_3 \vee \bar{x}_7) \wedge (\bar{x}_5 \vee x_3 \vee x_8) \wedge$$
$$(\bar{x}_6 \vee \bar{x}_1 \vee \bar{x}_5) \wedge (x_8 \vee \bar{x}_9 \vee x_3) \wedge (x_2 \vee x_1 \vee x_3) \wedge (\bar{x}_1 \vee x_8 \vee x_4) \wedge$$
$$(\bar{x}_9 \vee \bar{x}_6 \vee x_8) \wedge (x_8 \vee x_3 \vee \bar{x}_9) \wedge (x_9 \vee \bar{x}_3 \vee x_8) \wedge (x_6 \vee \bar{x}_9 \vee x_5) \wedge$$
$$(x_2 \vee \bar{x}_3 \vee \bar{x}_8) \wedge (x_8 \vee \bar{x}_6 \vee \bar{x}_3) \wedge (x_8 \vee \bar{x}_3 \vee \bar{x}_1) \wedge (\bar{x}_8 \vee x_6 \vee \bar{x}_2) \wedge$$
$$(x_7 \vee x_9 \vee \bar{x}_2) \wedge (x_8 \vee \bar{x}_9 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_9 \vee x_4) \wedge (x_8 \vee x_1 \vee \bar{x}_2) \wedge$$
$$(x_3 \vee \bar{x}_4 \vee \bar{x}_6) \wedge (\bar{x}_1 \vee \bar{x}_7 \vee x_5) \wedge (\bar{x}_7 \vee x_1 \vee x_6) \wedge (\bar{x}_5 \vee x_4 \vee \bar{x}_6) \wedge$$
$$(\bar{x}_4 \vee x_9 \vee \bar{x}_8) \wedge (x_2 \vee x_9 \vee x_1) \wedge (x_5 \vee \bar{x}_7 \vee x_1) \wedge (\bar{x}_7 \vee \bar{x}_9 \vee \bar{x}_6) \wedge$$
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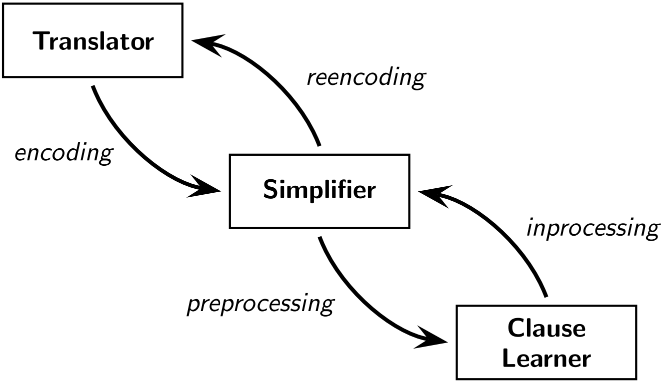
Does there exist an assignment satisfying all clauses?

Interaction between different solving approaches



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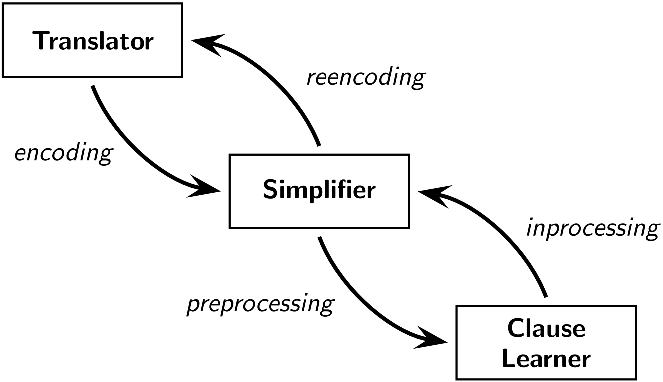

It all comes down to adding and removing redundant clauses

Redundant clauses

A clause is redundant with respect to a formula if adding it to the formula preserves satisfiability.

- For unsatisfiable formulas, all clauses can be added, including the empty clause ( ).

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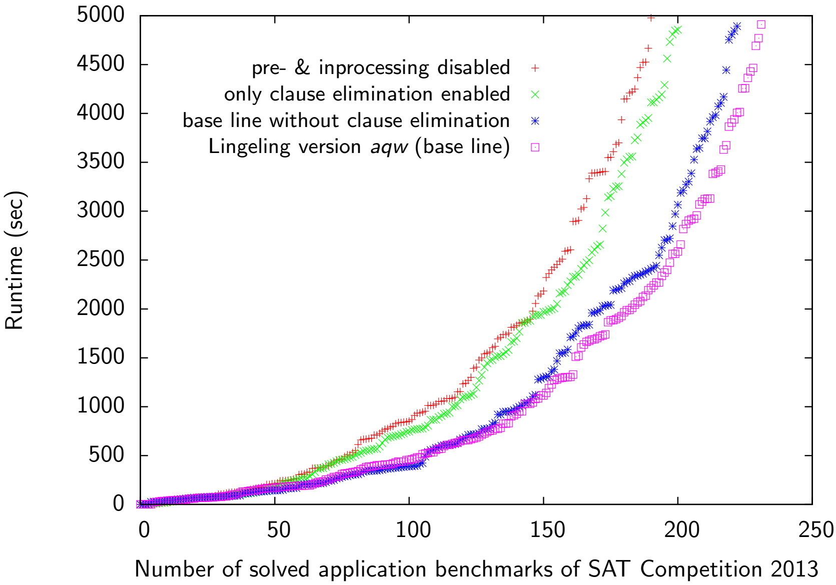
Challenge regarding redundant clauses:

- How to check redundancy in polynomial time?
- Ideally find redundant clauses in linear time

Preprocessing and Inprocessing in Practice



Outline

- Subsumption
- Variable Elimination
- Bounded Variable Addition
- Blocked Clause Elimination
- Hyper Binary Resolution
- Unhiding Redundancy

Subsumption

Tautologies and Subsumption

Definition (Tautology)

A clause C is a tautology if its contains two complementary literals $l, \bar{l}$.

Example

The clause $(a \vee b \vee \bar{b})$ *is a tautology.*

Definition (Subsumption)

Clause C subsumes clause D if and only if $C \subset D$.

Example

The clause $(a \vee b)$ *subsumes clause* $(a \vee b \vee \bar{c})$.

Self-Subsuming Resolution

Self-Subsuming Resolution

$$\frac{C \vee l \qquad D \vee \bar{l}}{D} \quad C \subseteq D \qquad \frac{(a \vee b \vee l) \qquad (a \vee b \vee c \vee \bar{l})}{(a \vee b \vee c)}$$

resolvent D subsumes second antecedent $D \vee \bar{l}$

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Example

Assume a CNF contains both antecedents
$\ldots (a \vee b \vee l)(a \vee b \vee c \vee \bar{l}) \ldots$

If D is added, then $D \vee \bar{l}$ can be removed

which in essence removes $\bar{l}$ from $D \vee \bar{l}$
$\ldots (a \vee b \vee l)(a \vee b \vee c \quad) \ldots$

Initially in the SATeLite preprocessor, [EenBiere'07]
now common in most solvers (i.e., as pre- and inprocessing)

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resolvent D subsumes second antecedent $D \vee \bar{l}$

Example: Remove literals using self-subsumption

$$\begin{array}{l}
(a \vee b \vee c) \ \wedge \ (\bar{a} \vee b \vee c) \ \wedge \\
(\bar{a} \vee b \vee \bar{c}) \ \wedge \ (a \vee \bar{b} \vee c) \ \wedge \\
(\bar{a} \vee \bar{b} \vee d) \ \wedge \ (\bar{a} \vee \bar{b} \vee \bar{d}) \ \wedge \\
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(\bar{a} \vee \bar{b}) & \wedge & (\bar{a} \vee \bar{b} \vee \bar{d}) \wedge \\
(a \vee \bar{c}) & \wedge & (a \vee \bar{c} \vee \bar{d})
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Implementing Subsumtion

Definition (Subsumption)

Clause C subsumes clause D if and only if $C \subset D$.

Example

The clause $(a \vee b)$ *subsumes clause* $(a \vee b \vee \bar{c})$.

Forward Subsumption

for each clause C in formula F **do**
  if C is subsumed by a clause D in $F \setminus C$ **then**
    remove C from F

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Backward Subsumption

for each clause C in formula F **do**
  remove all clauses D in F that are subsumed by C

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for each clause C in formula F **do**
  if C is subsumed by a clause D in $F \setminus C$ **then**
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Backward Subsumption

for each clause C in formula F **do**
  pick a literal l in C
  remove all clauses D in F_l that are subsumed by C

Variable Elimination

Variable Elimination [DavisPutnam'60]

Definition (Resolution)

Given two clauses $C = (x \vee a_1 \vee \cdots \vee a_i)$ and $D = (\bar{x} \vee b_1 \vee \cdots \vee b_j)$, the *resolvent* of C and D on variable x (denoted by $C \otimes_x D$) is $(a_1 \vee \cdots \vee a_i \vee b_1 \vee \cdots \vee b_j)$

Resolution on sets of clauses F_x and $F_{\bar{x}}$ (denoted by $F_x \otimes_x F_{\bar{x}}$) generates all (non-tautological) resolvents of $C \in F_x$ and $D \in F_{\bar{x}}$.

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Given a CNF formula F, *variable elimination* (or DP resolution) removes a variable x by replacing F_x and $F_{\bar{x}}$ by $F_x \otimes_x F_{\bar{x}}$

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Proof procedure [DavisPutnam60]

VE is a complete proof procedure. Applying VE until fixpoint results in either the empty formula (satisfiable) or empty clause (unsatisfiable)

Example VE by clause distribution [DavisPutnam'60]

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Example of clause distribution

	F_x		
	$(x \vee c)$	$(x \vee \bar{d})$	$(x \vee \bar{a} \vee \bar{b})$
$F_{\bar{x}}$: $(\bar{x} \vee a)$	$(a \vee c)$	$(a \vee d)$	$(a \vee \bar{a} \vee \bar{b})$
$(\bar{x} \vee b)$	$(b \vee c)$	$(b \vee d)$	$(b \vee \bar{a} \vee \bar{b})$
$(\bar{x} \vee \bar{e} \vee f)$	$(c \vee \bar{e} \vee f)$	$(d \vee \bar{e} \vee f)$	$(\bar{a} \vee \bar{b} \vee \bar{e} \vee f)$

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		F_x		
		$(x \lor c)$	$(x \lor \bar{d})$	$(x \lor \bar{a} \lor \bar{b})$
$F_{\bar{x}}$	$(\bar{x} \lor a)$	$(a \lor c)$	$(a \lor d)$	~~$(a \lor \bar{a} \lor \bar{b})$~~
	$(\bar{x} \lor b)$	$(b \lor c)$	$(b \lor d)$	~~$(b \lor \bar{a} \lor \bar{b})$~~
	$(\bar{x} \lor \bar{e} \lor f)$	$(c \lor \bar{e} \lor f)$	$(d \lor \bar{e} \lor f)$	$(\bar{a} \lor \bar{b} \lor \bar{e} \lor f)$

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$F_{\bar{x}}$: $(\bar{x} \vee \bar{e} \vee f)$	$(c \vee \bar{e} \vee f)$	$(d \vee \bar{e} \vee f)$	$(\bar{a} \vee \bar{b} \vee \bar{e} \vee f)$

In the example: $|F_x \otimes F_{\bar{x}}| > |F_x| + |F_{\bar{x}}|$

Exponential growth of clauses in general

VE by substitution [EenBiere07]

General idea

Detect gates (or definitions) $x = \mathrm{GATE}(a_1, \ldots, a_n)$ in the formula and use them to reduce the number of added clauses

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Possible gates

gate	G_x	$G_{\bar{x}}$
$\mathrm{AND}(a_1, \ldots, a_n)$	$(x \vee \bar{a}_1 \vee \cdots \vee \bar{a}_n)$	$(\bar{x} \vee a_1), \ldots, (\bar{x} \vee a_n)$
$\mathrm{OR}(a_1, \ldots, a_n)$	$(x \vee \bar{a}_1), \ldots, (x \vee \bar{a}_n)$	$(\bar{x} \vee a_1 \vee \cdots \vee a_n)$
$\mathrm{ITE}(c, t, f)$	$(x \vee \bar{c} \vee \bar{t}), (x \vee c \vee \bar{f})$	$(\bar{x} \vee \bar{c} \vee t), (\bar{x} \vee c \vee f)$

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$\mathrm{ITE}(c, t, f)$	$(x \vee \bar{c} \vee \bar{t}), (x \vee c \vee \bar{f})$	$(\bar{x} \vee \bar{c} \vee t), (\bar{x} \vee c \vee f)$

Variable elimination by substitution [EenBiere07]

Let $R_x = F_x \setminus G_x$; $R_{\bar{x}} = F_{\bar{x}} \setminus G_{\bar{x}}$.

Replace $F_x \wedge F_{\bar{x}}$ by $G_x \otimes_x R_{\bar{x}} \wedge G_{\bar{x}} \otimes_x R_x$.

Always less than $F_x \otimes_x F_{\bar{x}}$!

VE by substitution [EenBiere'07]

Example of gate extraction: $x = \mathrm{AND}(a, b)$

$$F_x = (x \vee c) \wedge (x \vee \bar{d}) \wedge (x \vee \bar{a} \vee \bar{b})$$
$$F_{\bar{x}} = (\bar{x} \vee a) \wedge (\bar{x} \vee b) \wedge (\bar{x} \vee \bar{e} \vee f)$$

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Example of substitution

		R_x		G_x
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$G_{\bar{x}}$	$(\bar{x} \vee a)$	$(a \vee c)$	$(a \vee d)$	
	$(\bar{x} \vee b)$	$(b \vee c)$	$(b \vee d)$	
$R_{\bar{x}}$	$(\bar{x} \vee \bar{e} \vee f)$			$(\bar{a} \vee \bar{b} \vee \bar{e} \vee f)$

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$G_{\bar{x}}$	$(\bar{x} \lor a)$	$(a \lor c)$	$(a \lor d)$	
	$(\bar{x} \lor b)$	$(b \lor c)$	$(b \lor d)$	
$R_{\bar{x}}$	$(\bar{x} \lor \bar{e} \lor f)$			$(\bar{a} \lor \bar{b} \lor \bar{e} \lor f)$

using substitution: $|F_x \otimes F_{\bar{x}}| < |F_x| + |F_{\bar{x}}|$

Bounded Variable Addition

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Main Idea

Given a CNF formula F, can we construct a (semi)logically equivalent F' by introducing a new variable $x \notin VAR(F)$ such that $|F'| < |F|$?

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Reverse of Variable Elimination

For example, replace the clauses

$$\begin{array}{ccc} (a \vee c) & (a \vee d) & \\ (b \vee c) & (b \vee d) & \\ (c \vee \bar{e} \vee f) & (d \vee \bar{e} \vee f) & (\bar{a} \vee \bar{b} \vee \bar{e} \vee f) \end{array}$$

by

$$\begin{array}{ccc} (\bar{x} \vee a) & (\bar{x} \vee b) & (\bar{x} \vee \bar{e} \vee f) \\ (x \vee c) & (x \vee d) & (x \vee \bar{a} \vee \bar{b}) \end{array}$$

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Challenge: how to find suitable patterns for replacement?

Factoring Out Subclauses

Example

Replace

$$(a \vee b \vee c \vee d) \quad (a \vee b \vee c \vee e) \quad (a \vee b \vee c \vee f)$$

by

$$(x \vee d) \quad (x \vee e) \quad (x \vee f) \quad (\bar{x} \vee a \vee b \vee c)$$

adds 1 variable and 1 clause *reduces* number of literals by 2

Not compatible with VE, which would eliminate x immediately!

... *so this does not work* ...

Bounded Variable Addition

Example

Smallest pattern that is compatible: Replace

$$\begin{array}{ll} (a \vee d) & (a \vee e) \\ (b \vee d) & (b \vee e) \\ (c \vee d) & (c \vee e) \end{array}$$

by

$$\begin{array}{lll} (\bar{x} \vee a) & (\bar{x} \vee b) & (\bar{x} \vee c) \\ (x \vee d) & (x \vee e) & \end{array}$$

adds 1 variable

removes 1 clause

Bounded Variable Addition

Possible Patterns

$$\begin{matrix} (X_1 \vee L_1) & \dots & (X_1 \vee L_k) \\ \vdots & & \vdots \\ (X_n \vee L_1) & \dots & (X_n \vee L_k) \end{matrix} \quad \equiv \quad \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{k} (X_i \vee L_j)$$

replaced by $\bigwedge_{i=1}^{n} (y \vee X_i) \wedge \bigwedge_{j=1}^{k} (\bar{y} \vee L_j)$

- Every k clauses share sets of literals L_j
- There are n sets of literals X_i that appear in clauses with L_j

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Bounded Variable Addition on AtMostOneZero (1)

Example encoding of AtMostOneZero $(x_1, x_2, \ldots, x_n)$

$$
\begin{aligned}
&(x_1 \vee x_2) \wedge (x_9 \vee x_{10}) \wedge (x_8 \vee x_{10}) \wedge (x_7 \vee x_{10}) \wedge (x_6 \vee x_{10}) \wedge \\
&(x_1 \vee x_3) \wedge (x_2 \vee x_3) \wedge (x_8 \vee x_9) \wedge (x_7 \vee x_9) \wedge (x_6 \vee x_9) \wedge \\
&(x_1 \vee x_4) \wedge (x_2 \vee x_4) \wedge (x_3 \vee x_4) \wedge (x_7 \vee x_8) \wedge (x_6 \vee x_8) \wedge \\
&(x_1 \vee x_5) \wedge (x_2 \vee x_5) \wedge (x_3 \vee x_5) \wedge (x_4 \vee x_5) \wedge (x_6 \vee x_7) \wedge \\
&(x_1 \vee x_6) \wedge (x_2 \vee x_6) \wedge (x_3 \vee x_6) \wedge (x_4 \vee x_6) \wedge (x_5 \vee x_6) \wedge \\
&(x_1 \vee x_7) \wedge (x_2 \vee x_7) \wedge (x_3 \vee x_7) \wedge (x_4 \vee x_7) \wedge (x_5 \vee x_7) \wedge \\
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&(x_1 \vee x_{10}) \wedge (x_2 \vee x_{10}) \wedge (x_3 \vee x_{10}) \wedge (x_4 \vee x_{10}) \wedge (x_5 \vee x_{10})
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&(x_1 \vee x_5) \wedge (x_2 \vee x_5) \wedge (x_3 \vee x_5) \wedge (x_4 \vee x_5) \wedge (x_6 \vee x_7) \wedge \\
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\end{aligned}
$$

Replace $(x_i \vee x_j)$ with $i \in \{1..5\}, j \in \{6..10\}$ by $(x_i \vee y)$, $(x_j \vee \bar{y})$

Bounded Variable Addition on AtMostOneZero (2)

Example encoding of AtMostOneZero $(x_1, x_2, \ldots, x_n)$

$$
\begin{array}{l}
(x_1 \vee x_2) \wedge (x_9 \vee x_{10}) \wedge (x_8 \vee x_{10}) \wedge (x_7 \vee x_{10}) \wedge (x_6 \vee x_{10}) \wedge \\
(x_1 \vee x_3) \wedge (x_2 \vee x_3) \wedge (x_8 \vee x_9) \wedge (x_7 \vee x_9) \wedge (x_6 \vee x_9) \wedge \\
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(x_1 \vee y) \wedge (x_2 \vee y) \wedge (x_3 \vee y) \wedge (x_4 \vee y) \wedge (x_5 \vee y) \wedge \\
(x_6 \vee \bar{y}) \wedge (x_7 \vee \bar{y}) \wedge (x_8 \vee \bar{y}) \wedge (x_9 \vee \bar{y}) \wedge (x_{10} \vee \bar{y})
\end{array}
$$

Bounded Variable Addition on AtMostOneZero (2)

Example encoding of AtMostOneZero $(x_1, x_2, \ldots, x_n)$

$$
\begin{array}{l}
(x_1 \lor x_2) \land (x_9 \lor x_{10}) \land (x_8 \lor x_{10}) \land (x_7 \lor x_{10}) \land (x_6 \lor x_{10}) \land \\
(x_1 \lor x_3) \land (x_2 \lor x_3) \land (x_8 \lor x_9) \land (x_7 \lor x_9) \land (x_6 \lor x_9) \land \\
(x_1 \lor x_4) \land (x_2 \lor x_4) \land (x_3 \lor x_4) \land (x_7 \lor x_8) \land (x_6 \lor x_8) \land \\
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\end{array}
$$

Replace matched pattern

$$
\begin{array}{l}
(x_1 \lor z) \land (x_2 \lor z) \land (x_3 \lor z) \land \\
(x_4 \lor \bar{z}) \land (x_5 \lor \bar{z}) \land (y \lor \bar{z})
\end{array}
$$

Bounded Variable Addition on AtMostOneZero (3)

Example encoding of AtMostOneZero $(x_1, x_2, \ldots, x_n)$

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&(x_4 \lor \bar{z}) \land (x_5 \lor \bar{z}) \land (y \lor \bar{z}) \land (x_4 \lor x_5) \land (x_6 \lor x_7) \land \\
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\end{aligned}
$$

Bounded Variable Addition on AtMostOneZero (3)

Example encoding of AtMostOneZero $(x_1, x_2, \ldots, x_n)$

$$(x_1 \vee x_2) \wedge (x_9 \vee x_{10}) \wedge (x_8 \vee x_{10}) \wedge (x_7 \vee x_{10}) \wedge (x_6 \vee x_{10}) \wedge$$
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Replace matched pattern

$$(x_6 \vee w) \wedge (x_7 \vee w) \wedge (x_8 \vee w) \wedge$$
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Blocked Clause Elimination

Blocked Clauses [Kullmann'99]

Definition (Blocking literal)

A literal l in a clause C of a CNF F blocks C w.r.t. F if for every clause $D \in F_{\bar{l}}$, the resolvent $(C \setminus \{l\}) \cup (D \setminus \{\bar{l}\})$ obtained from resolving C and D on l is a tautology.

With respect to a fixed CNF and its clauses we have:

Definition (Blocked clause)

A clause is blocked if it contains a literal that blocks it.

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Example

Consider the formula $(a \vee b) \wedge (a \vee \bar{b} \vee \bar{c}) \wedge (\bar{a} \vee c)$.
First clause is not blocked.
Second clause is blocked by both a *and* $\bar{c}$.
Third clause is blocked by c

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First clause is not blocked.
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Proposition

Removal of an arbitrary blocked clause preserves satisfiability.

Blocked Clause Elimination (BCE)

Definition (BCE)

While there is a blocked clause C in a CNF F, remove C from F.

Example

Consider $(a \vee b) \wedge (a \vee \bar{b} \vee \bar{c}) \wedge (\bar{a} \vee c)$*.*
After removing either $(a \vee \bar{b} \vee \bar{c})$ *or* $(\bar{a} \vee c)$*, the clause* $(a \vee b)$ *becomes blocked (no clause with either* $\bar{b}$ *or* $\bar{a}$*).*

An extreme case in which BCE *removes all clauses!*

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An extreme case in which BCE *removes all clauses!*

Proposition

BCE is confluent, i.e., has a unique fixpoint

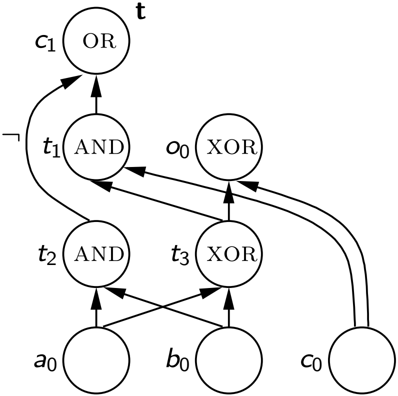
- Blocked clauses stay blocked w.r.t. removal

BCE very effective on circuits [JärvisaloBiereHeule'10]

BCE converts the Tseitin encoding to Plaisted Greenbaum

BCE simulates Pure literal elimination, Cone of influence and much more

Example of circuit simplification by BCE on Tseitin encoding



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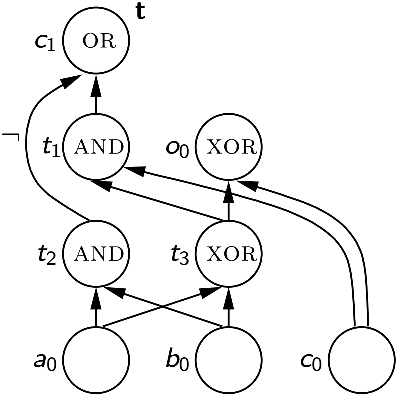

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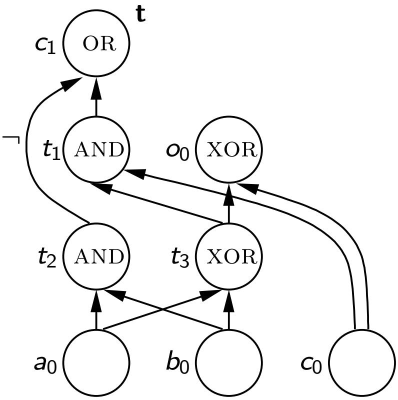

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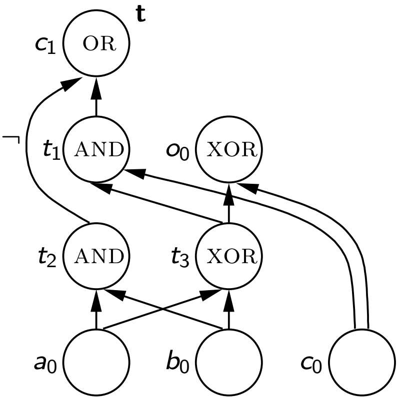

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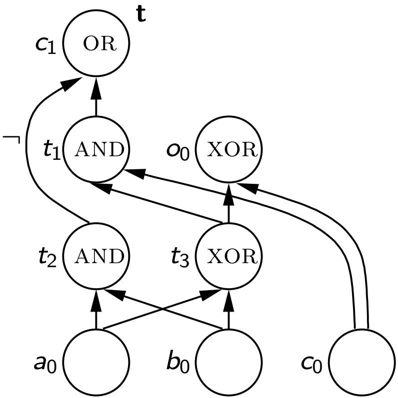

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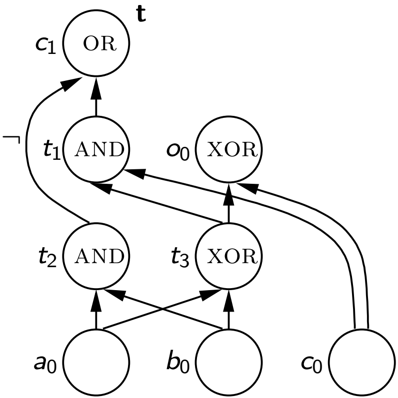

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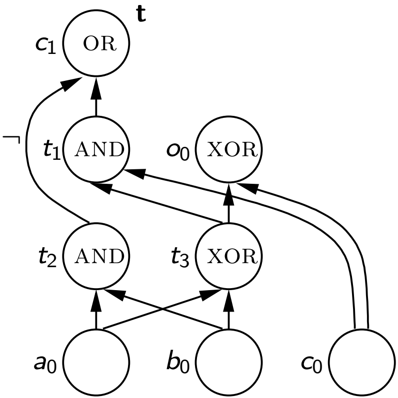

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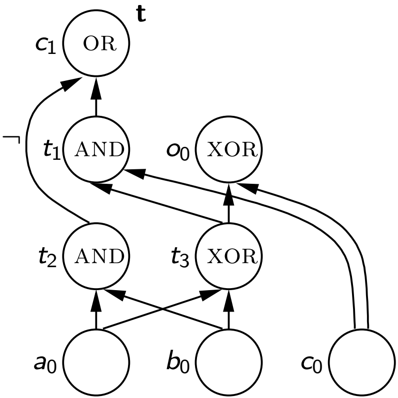

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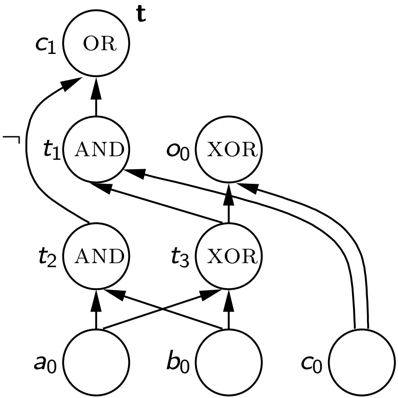

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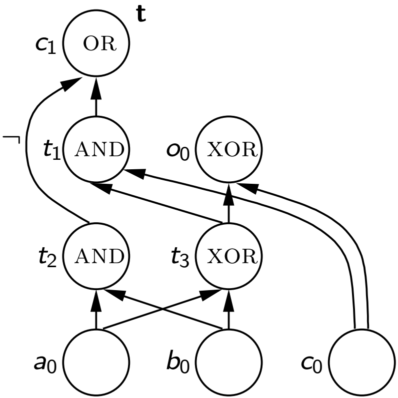

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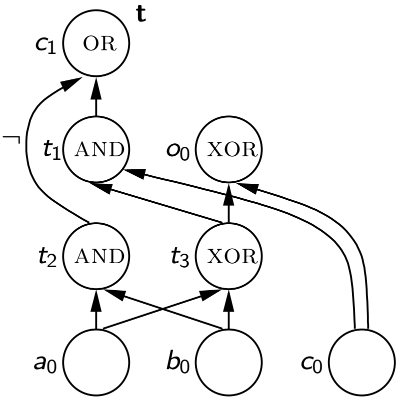

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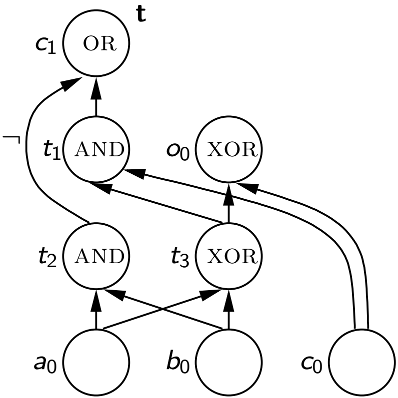
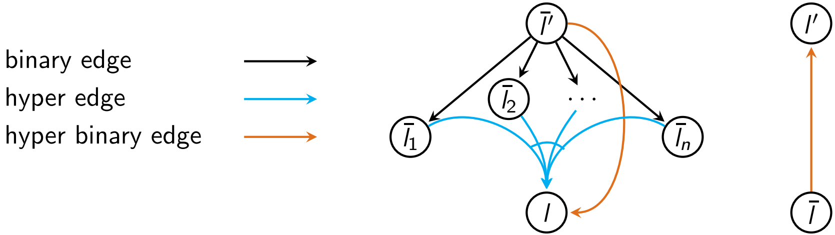


Hyper Binary Resolution

Hyper Binary Resolution [Bacchus-AAAI02]

Definition (Hyper Binary Resolution Rule)

$$\frac{(l \vee l_1 \vee l_2 \vee \cdots \vee l_n) \quad (\bar{l}_1 \vee l') \quad (\bar{l}_2 \vee l') \quad \ldots \quad (\bar{l}_n \vee l')}{(l \vee l')}$$



Hyper Binary Resolution Rule:

- combines multiple resolution steps into one
- uses one n-ary clauses and multiple binary clauses
- special case hyper unary resolution where $l = l'$

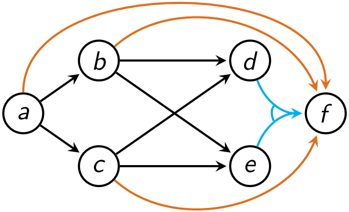
Hyper Binary Resolution (HBR)

Definition (Hyper Binary Resolution)

Apply the hyper binary resolution rule until fixpoint

Example

Consider
$(\bar{a} \vee b) \wedge (\bar{a} \vee c) \wedge (\bar{b} \vee d) \wedge (\bar{b} \vee e) \wedge (\bar{c} \vee d) \wedge (\bar{c} \vee e) \wedge (\bar{d} \vee \bar{e} \vee f)$.



hyper binary resolvents:
$(\bar{a} \vee f), (\bar{b} \vee f), (\bar{c} \vee f)$

HBR is confluent, i.e., has a unique fixpoint

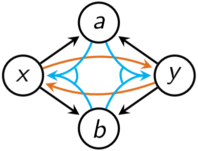
Structural Hashing of AND-gates via HBR

gate g	$g \Rightarrow f(g_1, \dots g_n)$ "positive"	$g \Leftarrow f(g_1, \dots g_n)$ "negative"
$g := \text{OR}(g_1, \dots, g_n)$	$(\bar{g} \vee g_1 \vee \dots \vee g_n)$	$(g \vee \bar{g_1}),\dots,(g \vee \bar{g_n})$
$g := \text{AND}(g_1, \dots, g_n)$	$(\bar{g} \vee g_1),\dots,(\bar{g} \vee g_n)$	$(g \vee \bar{g_1} \vee \dots \vee \bar{g_n})$
$g := \text{XOR}(g_1, g_2)$	$(\bar{g} \vee \bar{g_1} \vee \bar{g_2}), (\bar{g} \vee g_1 \vee g_2)$	$(g \vee \bar{g_1} \vee g_2), (g \vee g_1 \vee \bar{g_2})$
$g := \text{ITE}(g_1, g_2, g_3)$	$(\bar{g} \vee \bar{g_1} \vee g_2), (\bar{g} \vee g_1 \vee g_3)$	$(g \vee \bar{g_1} \vee \bar{g_2}), (g \vee g_1 \vee \bar{g_3})$

Definition (Structural Hashing of AND-gates)

Given a Boolean circuit with two equivalent gates, merge the gates.

Example



x = AND(a,b) : $(\bar{x} \vee a) \wedge (\bar{x} \vee b) \wedge (x \vee \bar{a} \vee \bar{b})$

y = AND(a,b) : $(\bar{y} \vee a) \wedge (\bar{y} \vee b) \wedge (y \vee \bar{a} \vee \bar{b})$

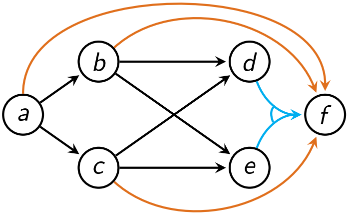
the two HBRs $(\bar{x} \vee y)$ and $(x \vee \bar{y})$ express that $x = y$

Non-transitive Hyper Binary Resolution (NHBR)

A problem with classic HBR is that it adds many transitive binary clauses

Example

Consider
$(\bar{a} \vee b) \wedge (\bar{a} \vee c) \wedge (\bar{b} \vee d) \wedge (\bar{b} \vee e) \wedge (\bar{c} \vee d) \wedge (\bar{c} \vee e) \wedge (\bar{d} \vee \bar{e} \vee f)$.



adding $(\bar{b} \vee f)$ or $(\bar{c} \vee f)$
makes $(\bar{a} \vee f)$ transitive

Solution [HeuleJärvisaloBiere 2013]

Add only non-transitive hyper binary resolvents
Can be implemented using an alternative unit propagation style

Space Complexity of NHBR: Quadratic

Question regarding complexity [Biere 2009]

- Are there formulas where the transitively reduced hyper binary resolution closure is quadratic in size w.r.t. to the size of the original?
- where size $=$ #clauses or size $=$ #literals or size $=$ #variables

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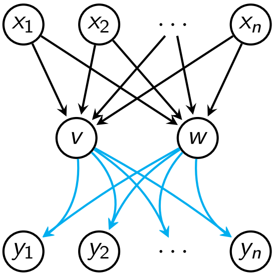
Yes!

Consider the formula $F_n = \bigwedge_{1 \leq i \leq n} \left((\bar{x}_i \vee v) \wedge (\bar{x}_i \vee w) \wedge (\bar{v} \vee \bar{w} \vee y_i) \right)$



#variables: $2n + 2$

#clauses: $3n$

#literals: $7n$

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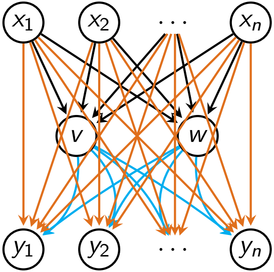

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#clauses: $3n$

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n^2 hyper binary resolvents:

$(\bar{x}_i \vee y_j)$ for $1 \leq i, j \leq n$

Unhiding Redundancy

Redundancy

Redundant clauses:

- Removal of $C \in F$ preserves unsatisfiability of F
- Assign all $l \in C$ to false and check for a conflict in $F \setminus \{C\}$

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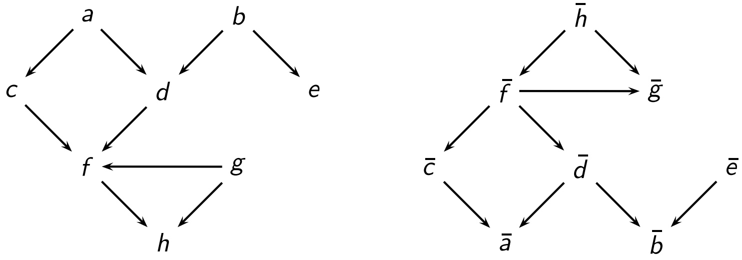
Redundancy elimination during pre- and in-processing

- Distillation [JinSomenzi2005]
- ReVivAl [PietteHamadiSaïs2008]
- Unhiding [HeuleJärvisaloBiere2011]

Unhide: Binary implication graph (BIG)

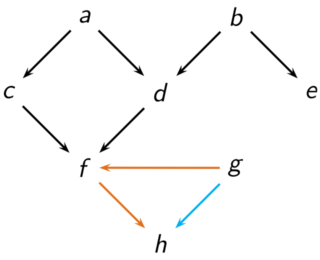
unhide: use the binary clauses to detect redundant clauses and literals

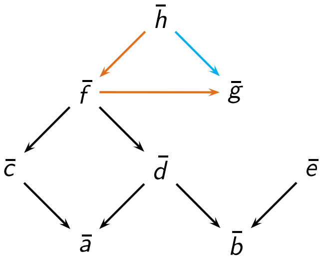


$$(\bar{a} \vee c) \wedge (\bar{a} \vee d) \wedge (\bar{b} \vee d) \wedge (\bar{b} \vee e) \wedge$$
$$(\bar{c} \vee f) \wedge (\bar{d} \vee f) \wedge (\bar{g} \vee f) \wedge (\bar{f} \vee h) \wedge$$
$$(\bar{g} \vee h) \wedge \underbrace{(\bar{a} \vee \bar{e} \vee h) \wedge (\bar{b} \vee \bar{c} \vee h) \wedge (a \vee b \vee c \vee d \vee e \vee f \vee g \vee h)}_{\text{non binary clauses}}$$

Unhide: Transitive reduction (TRD)

transitive reduction: remove shortcuts in the binary implication graph





$(\bar{a} \vee c) \wedge (\bar{a} \vee d) \wedge (\bar{b} \vee d) \wedge (\bar{b} \vee e) \wedge$

$(\bar{c} \vee f) \wedge (\bar{d} \vee f) \wedge (\bar{g} \vee f) \wedge (\bar{f} \vee h) \wedge$

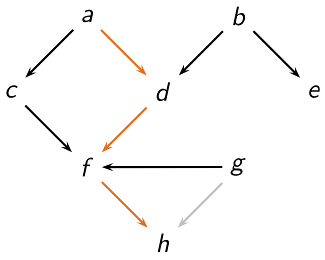
~~$(\bar{g} \vee h)$~~ $\wedge (\bar{a} \vee \bar{e} \vee h) \wedge (\bar{b} \vee \bar{c} \vee h) \wedge (a \vee b \vee c \vee d \vee e \vee f \vee g \vee h)$

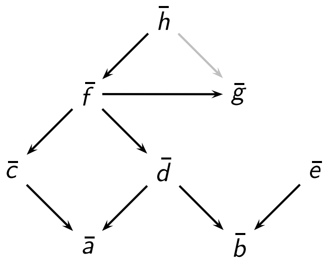
TRD

$g \rightarrow f \rightarrow h$

Unhide: Hidden tautology elimination (HTE) (1)

HTE removes clauses that are subsumed by an implication in BIG





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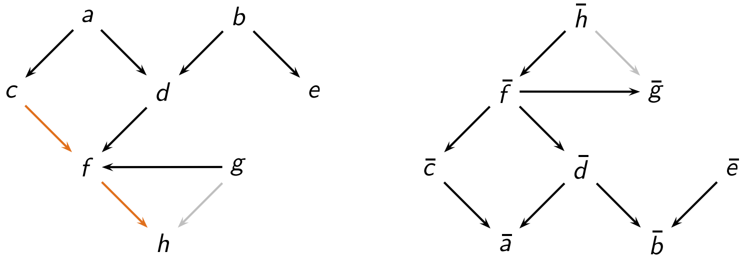
~~$(\bar{a} \vee \bar{e} \vee h)$~~ $\wedge (\bar{b} \vee \bar{c} \vee h) \wedge (a \vee b \vee c \vee d \vee e \vee f \vee g \vee h)$

HTE

$a \rightarrow d \rightarrow f \rightarrow h$

Unhide: Hidden tautology elimination (HTE) (2)

HTE removes clauses that are subsumed by an implication in BIG



$$(\bar{a} \lor c) \land (\bar{a} \lor d) \land (\bar{b} \lor d) \land (\bar{b} \lor e) \land$$
$$(\bar{c} \lor f) \land (\bar{d} \lor f) \land (\bar{g} \lor f) \land (\bar{f} \lor h) \land$$

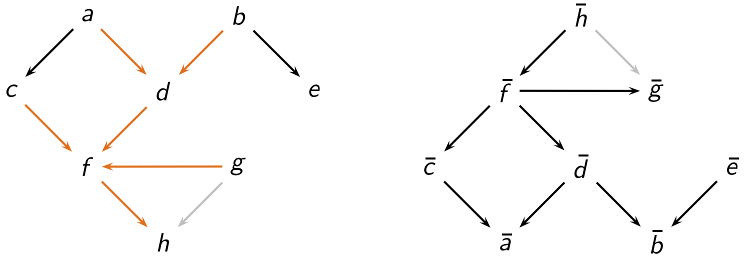
~~$(\bar{b} \lor \bar{c} \lor h)$~~ $\land (a \lor b \lor c \lor d \lor e \lor f \lor g \lor h)$

HTE

$c \to f \to h$

Unhide: Hidden literal elimination (HLE)

HLE removes literal using the implication in BIG



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$(\bar{c} \vee f) \wedge (\bar{d} \vee f) \wedge (\bar{g} \vee f) \wedge (\bar{f} \vee h) \wedge$

$(a \vee b \vee c \vee d \vee e \vee f \vee g \vee h)$

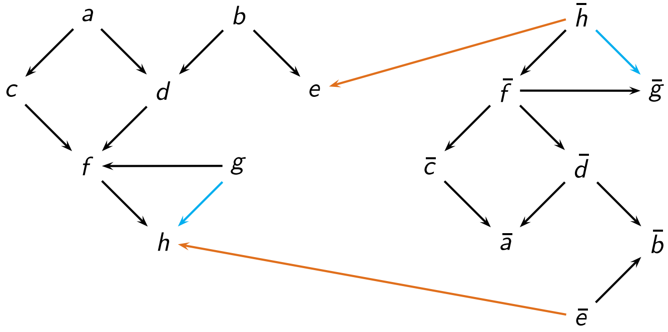
HLE

all but e imply h

also b implies e

Unhide: TRD + HTE + HLE

unhide: redundancy elimination removes and adds arcs from BIG(F)



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Conclusions

Many pre- or in-processing techniques in SAT solvers:

- (Self-)Subsumption
- Variable Elimination
- Blocked Clause Elimination
- Hyper Binary Resolution
- Bounded Variable Addition
- Equivalent Literal Substitution
- Failed Literal Elimination
- Autarky Reasoning
- ...

Preprocessing and Inprocessing

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THE UNIVERSITY OF TEXAS AT AUSTIN

Summer School on Formal Techniques, May 23, 2017