Piecewise Affine Models from Input-Output Data

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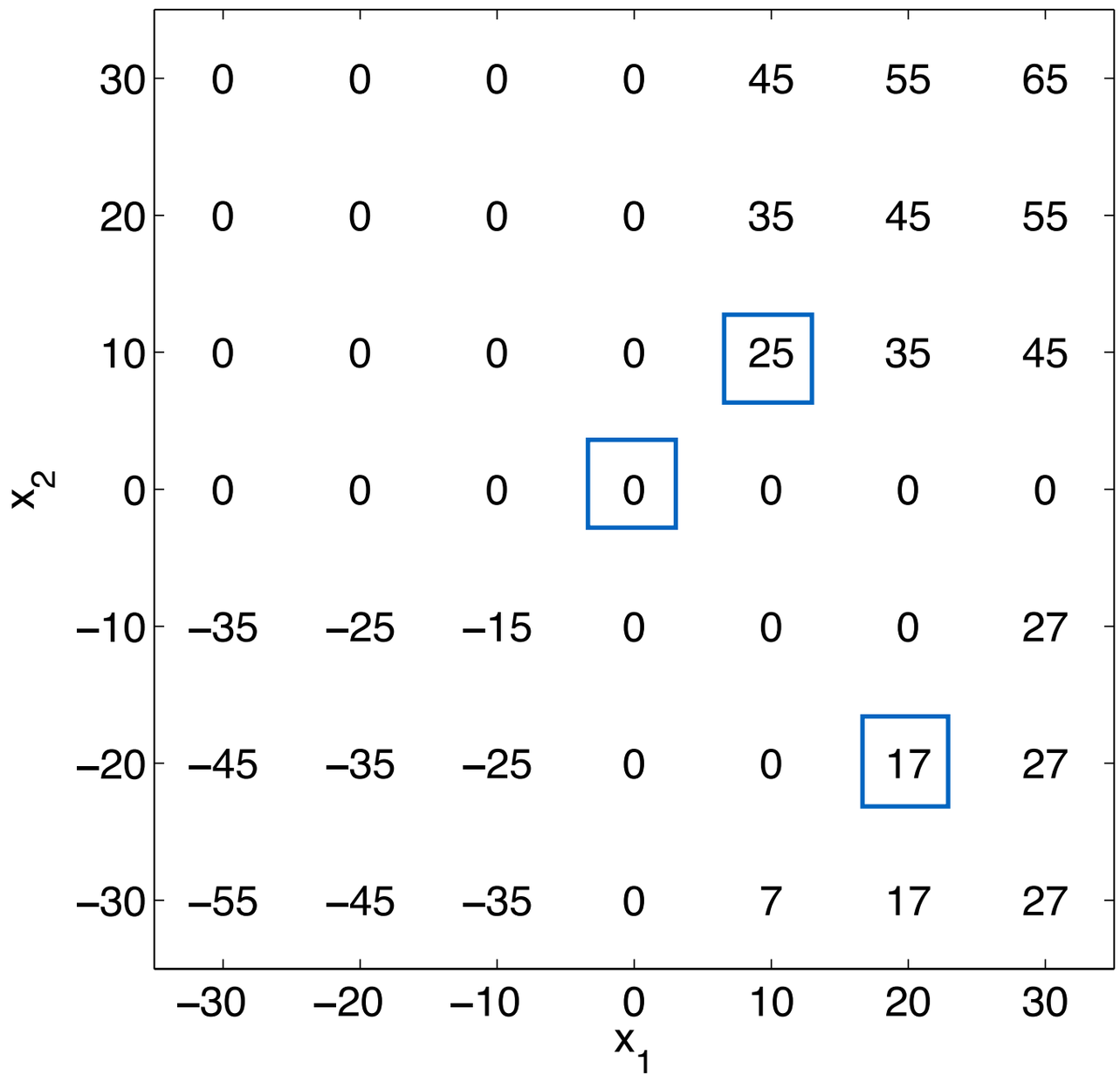
Problem

- To represent real valued Input-Output Data ***D*** succinctly using a piecewise linear function
- A piecewise linear function
  - Input domain partitioned into regions
  - Each region mapped to a linear function
  - Region represented by a guard condition

Example

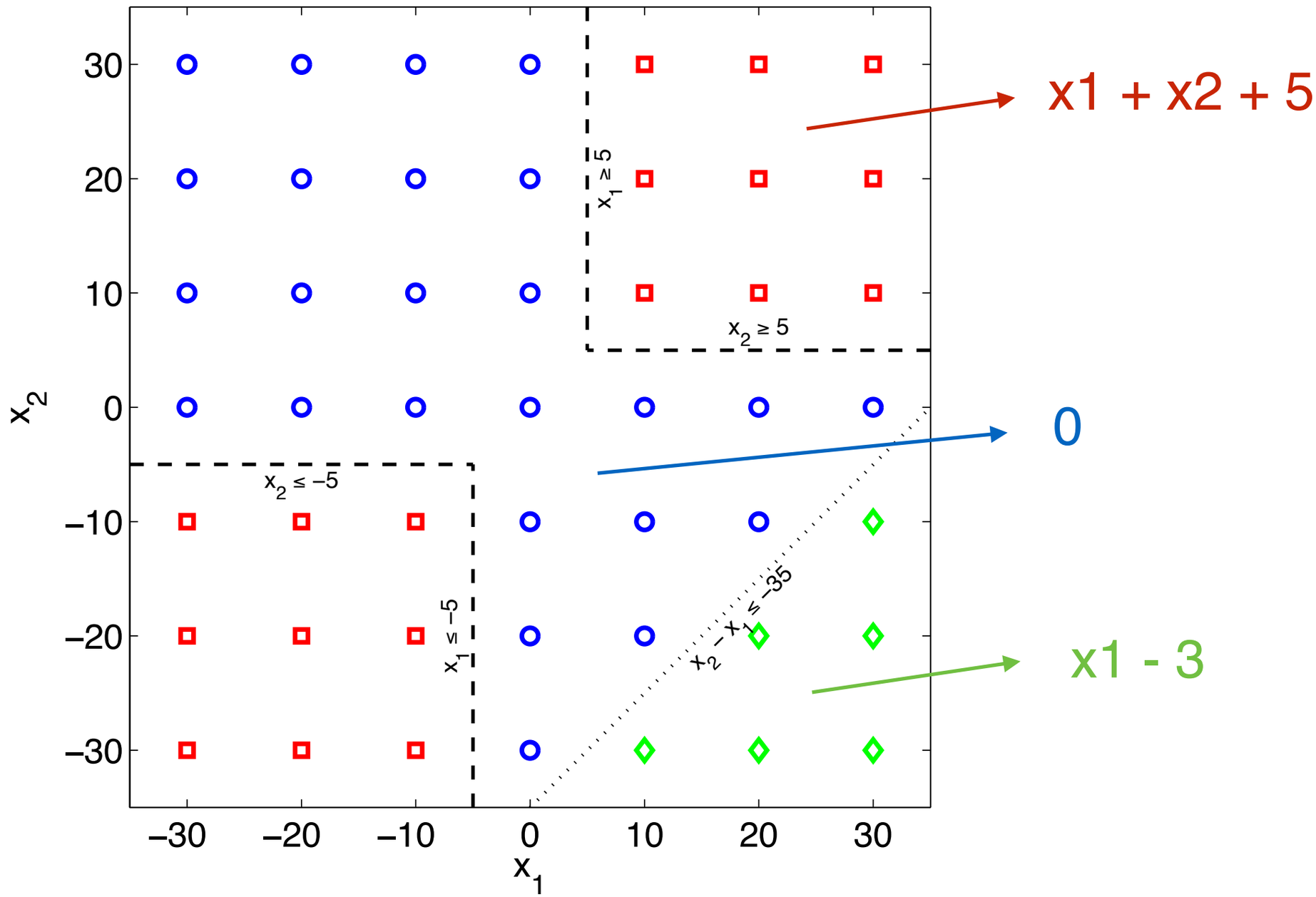


(10, 10) → 25

(0, 0) → 0

(20, -20) →17

Example


30
20
10
0
−10
−20
−30
x2
−30
−20
−10
0
10
20
30
x1
x1 ≥ 5
x2 ≥ 5
x2 ≤ −5
x1 ≤ −5
x2 − x1 ≤ −35
x1 + x2 + 5
0
x1 - 3

Example

- Piecewise Linear Function f:

f(x1, x2) =

if x2 − x1 ≤ −35 **then**

guard condition

x1 − 3

linear function

else if (x1 ≤ −5 ∧ x2 ≤ −5) ∨ (5 ≤ x1 ∧ 5 ≤ x2) **then**

x1 + x2 + 5

else

0

Solution

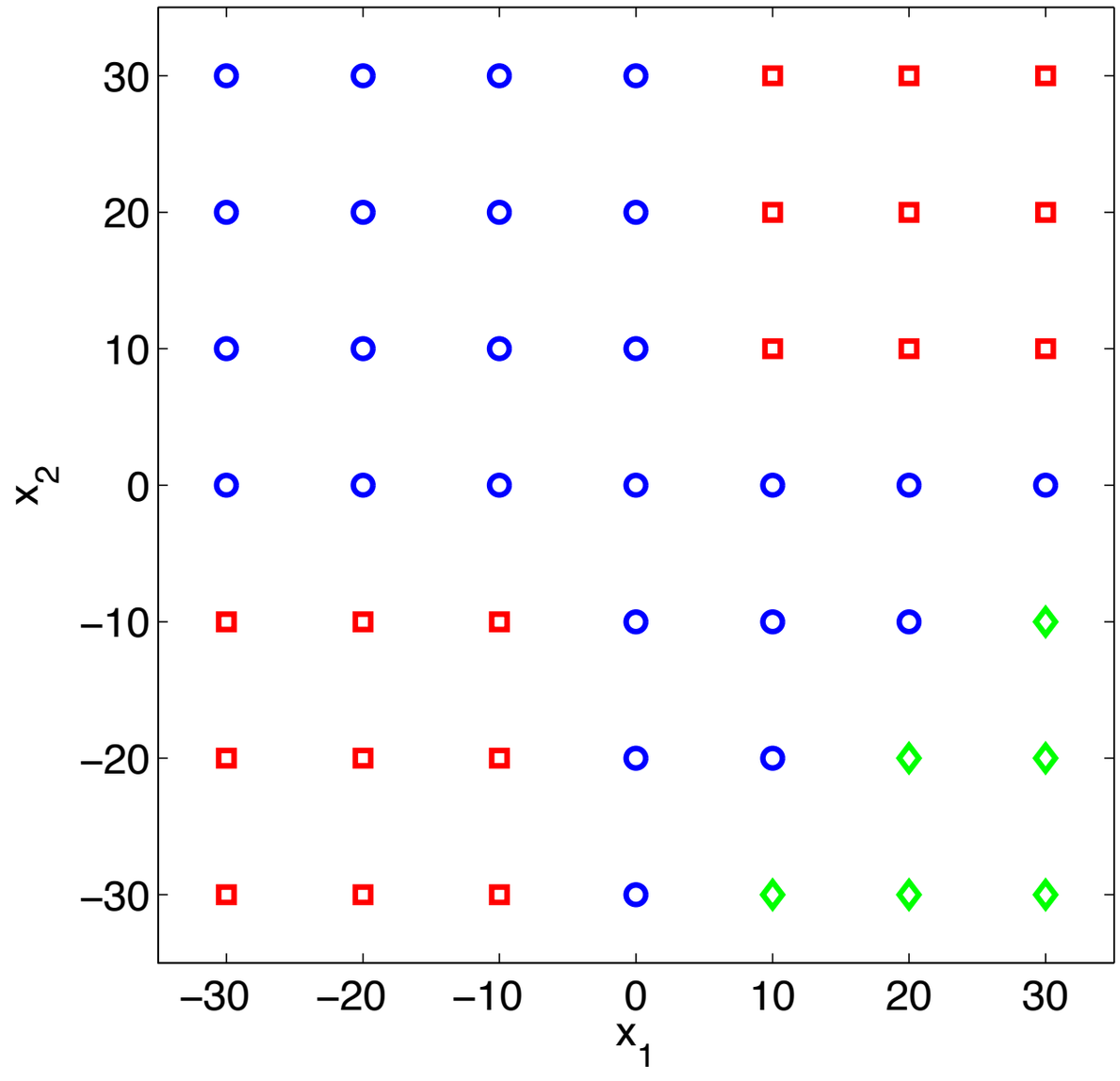
- In 2 phases
  - Find a set of *linear functions L*
    - Find *guard condition* for each linear function in *L*
- Optimal solutions to both problems are NP-Hard
  - Only best effort solution

Finding Linear Functions

- To find a set of linear functions *L* such that
  - each (x, y) in ***D*** is represented by a linear function in *L*
- Learn *L* iteratively
  - Pick a random point p in ***D***
  - Find a linear function that represents points in the neighborhood of p and add it to *L*
  - Remove the covered points from ***D*** and repeat

Finding Linear Functions

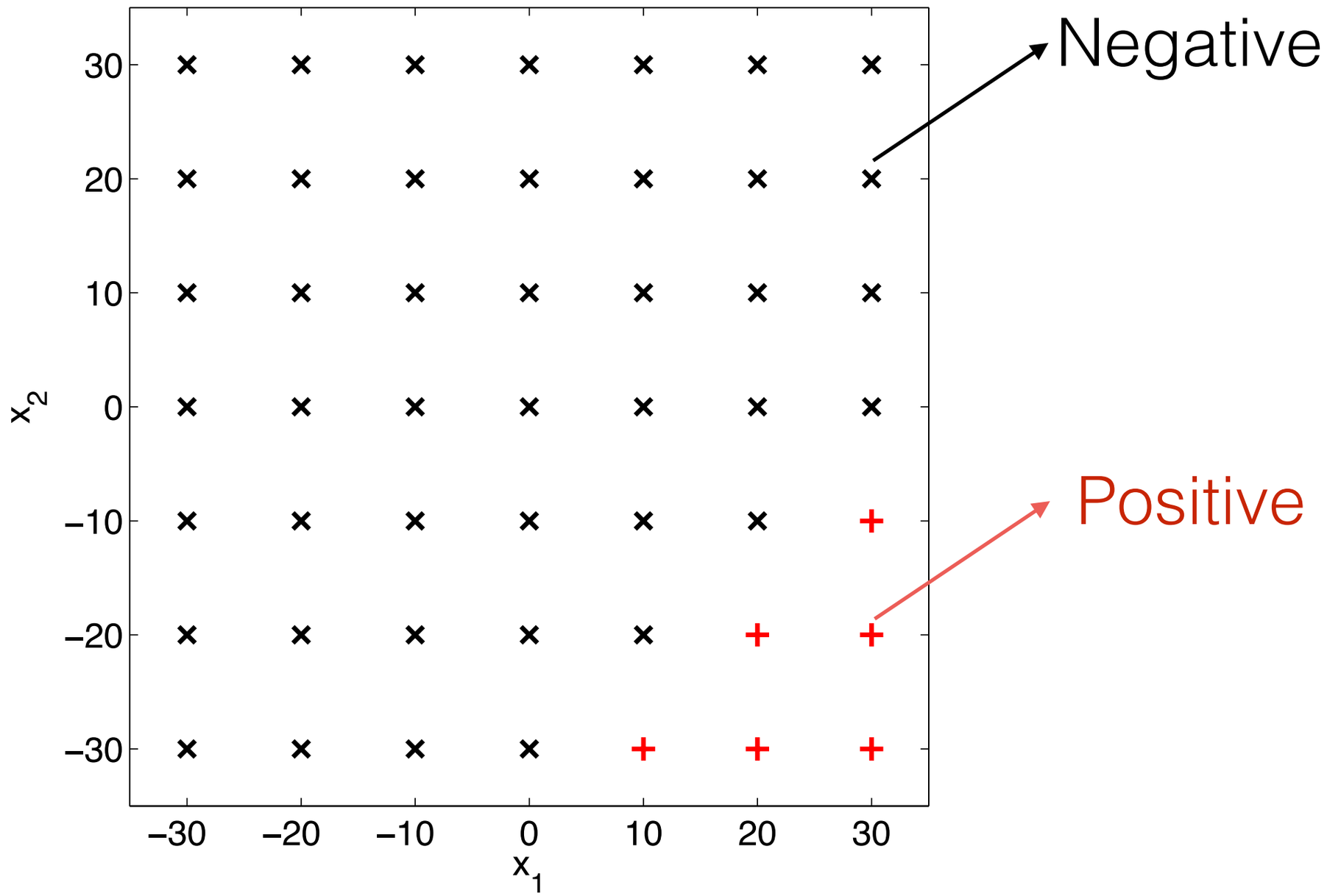
- Say 3 linear functions found for the example
  - l1: x1 - 3
  - l2: x1 + x2 + 5
  - l3: 0
- Every point in ***D*** represented by a linear function



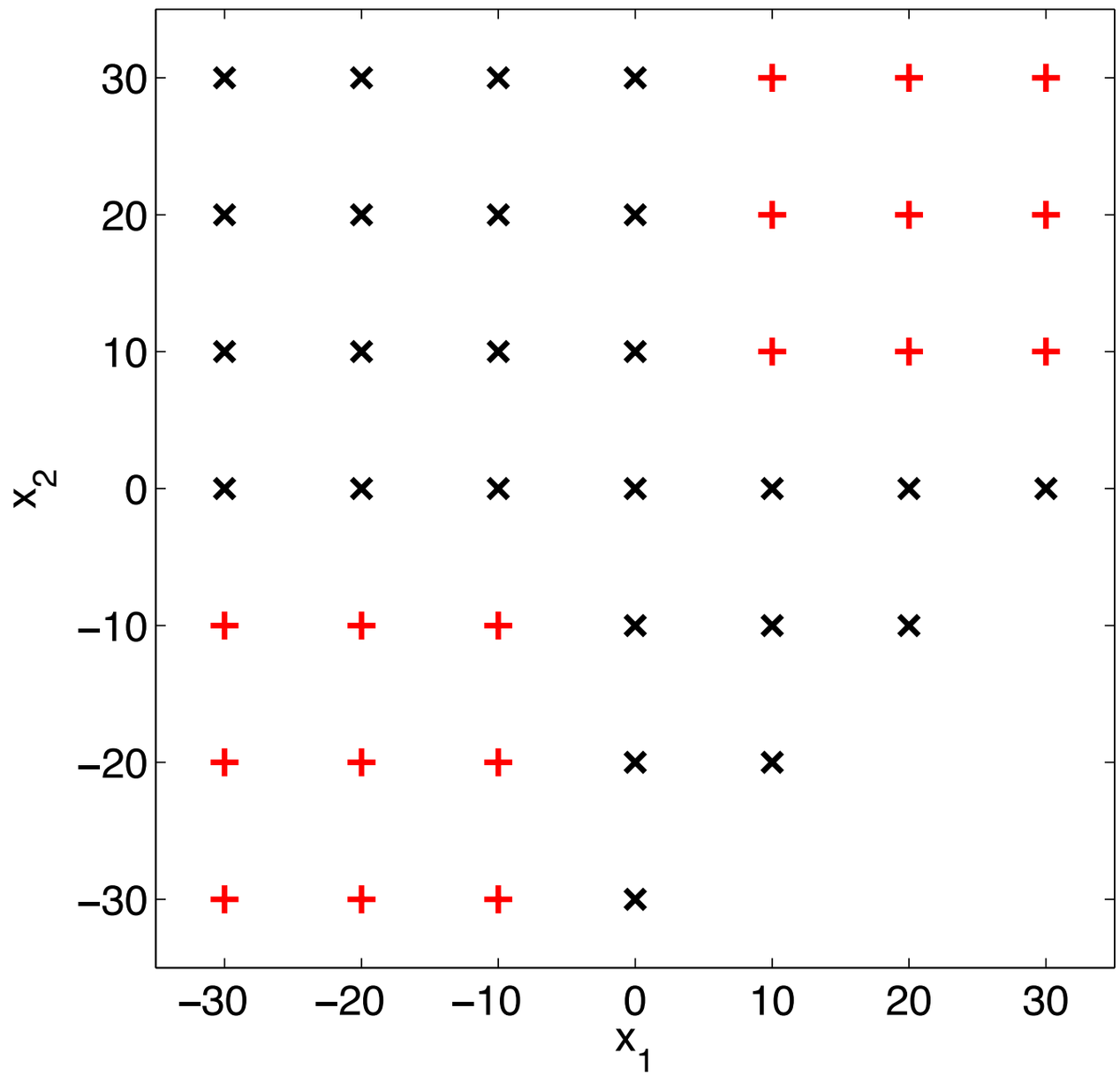
Finding Guards

- To find guard condition Φ for the linear function *l* in *L*
  - marks the region where *l* is defined
  - Φ is *true* on points represented by *l*, marked **positive**
    - Φ is *false* on points *not* represented by *l*, marked **negative**
    - Φ needs to separate **positive** and **negative** points
- Problem of learning a precise *binary classifier*
- Problem of learning *interpolant*

Guard for I1



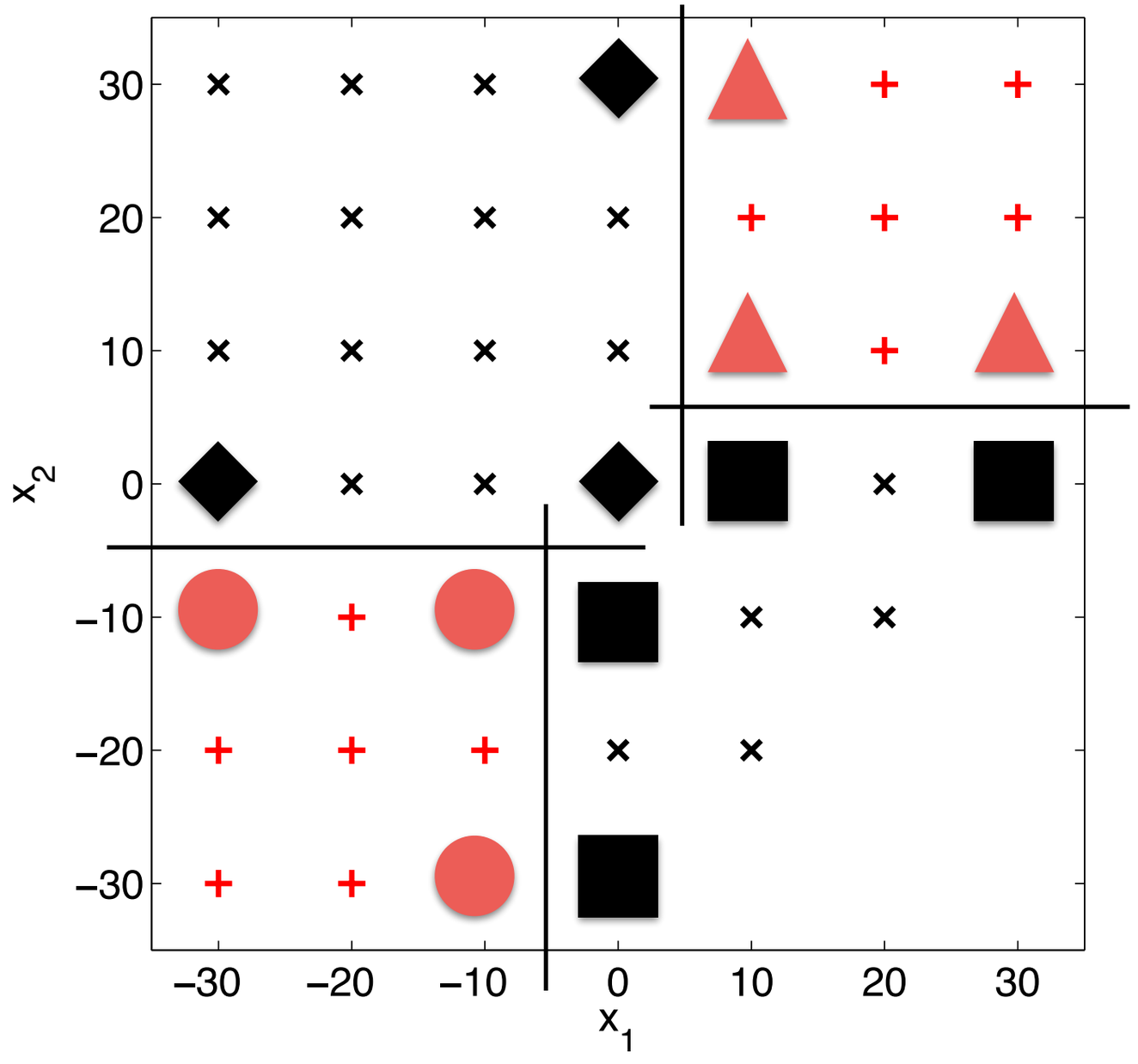
Guard for I2



Precise Classifiers

- Find *groups* of positive and negative points
- Each positive and negative group can be separated by a linear inequality
- Combine inequalities so that all positive groups can be separated from negative groups

Guard for I2



Precise Classifiers

- Use counter example guided approach to learn these groups
  - Start with a single positive and negative point as positive and negative group.
    - Iteratively update groups using counter examples until correct classifier is found
- Based on interpolant learning technique
  - “Beautiful Interpolants” by Albarghouthi et. al in CAV 2013.

Conclusion

- Presented a technique to represent input-output data using piecewise linear functions
- Combined use of machine learning and verification techniques
- One potential application in modeling hybrid systems
- Questions?